

# SUCCESSIVE FAILURES OF APPROACHABILITY

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ABSTRACT. Motivated by showing that in ZFC we cannot construct a special Aronszajn tree on some cardinal greater than  $\aleph_1$ , we produce a model in which the approachability property fails (hence there are no special Aronszajn trees) at all regular cardinals in the interval  $[\aleph_2, \aleph_{\omega^2+3}]$  and  $\aleph_{\omega^2}$  is strong limit.

## 1. INTRODUCTION

In the 1920's, König [11] proved that every tree of height  $\omega$  with finite levels has a cofinal branch. In the 1930's Aronszajn [12] showed that the analogous theorem for  $\omega_1$  fails. In particular he constructed a tree of height  $\omega_1$  whose levels are countable which has no cofinal branch. Such trees have come to be known as Aronszajn trees. The first Aronszajn tree is *special* in the sense that there is a function  $f : T \rightarrow \omega$  such that  $f(s) \neq f(t)$  whenever  $s$  is below  $t$  in  $T$ . This function  $f$  witnesses that  $T$  has no cofinal branch.

These two theorems provide a strong contrast between the combinatorial properties of  $\omega$  and  $\omega_1$ . König's theorem shows that  $\omega$  has a certain compactness property, while an Aronszajn tree is a canonical example of a noncompact object of size  $\omega_1$ . These properties admit straightforward generalizations to higher cardinals. We say that a regular cardinal  $\lambda$  has the *tree property* if it satisfies a higher analog of König's theorem. If  $\lambda$  does not have the tree property, then we call a counter example a  $\lambda$ -Aronszajn tree. The natural question is: Which cardinals carry Aronszajn trees?

A full answer to this question is connected to the phenomena of independence in set theory and large cardinals. The first evidence of this comes from a theorem of Specker [26] which shows that Aronszajn's construction can be generalized in the context of an instance of the generalized continuum hypothesis. In particular if  $\kappa^{<\kappa} = \kappa$ , then there is a special  $\kappa^+$ -Aronszajn tree (for the appropriate generalization of the notion of special). On the other hand, the tree property has a strong connection with the existence of large cardinals. We say that an uncountable cardinal is *weakly compact* if it satisfies a higher analog of infinite Ramsey's theorem. By theorems of Tarski and Erdős [6] and Monk and Scott [15], an uncountable cardinal  $\lambda$  is weakly compact if and only if it is inaccessible and has the tree property.

The invention of forcing provided method for proving the consistency of the tree property at accessible cardinals. An early theorem of Mitchell and Silver [14], shows that the tree property at  $\omega_2$  is equiconsistent with the existence of a weakly compact cardinal. So if the existence of a weakly compact cardinal is consistent, then it is impossible to construct an  $\omega_2$ -Aronszajn tree from the usual axioms of set theory. Moreover, the assumption that a weakly compact cardinal is consistent is necessary. This result gives an approach to resolving which cardinals have Aronszajn trees.

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The conjecture is that if the existence of enough large cardinals is consistent, then we cannot prove the statement that for some  $\lambda$  there is  $\lambda$ -Aronszajn tree. A weaker goal which captures many of the difficulties of this conjecture is to show that in ZFC one cannot prove that there is a cardinal  $\lambda$  which carries a *special*  $\lambda$ -Aronszajn tree.

It is this weaker goal that we address in this paper. We prove

**Theorem 1.1.** *Under suitable large cardinal hypotheses it is consistent that  $\aleph_{\omega^2}$  is strong limit and the approachability property fails for every regular cardinal in the interval  $[\aleph_2, \aleph_{\omega^2+3}]$ .*

As we will mention below, the failure of the approachability property at a cardinal  $\lambda$  is a strengthening of the nonexistence of special  $\lambda$ -Aronszajn trees. So the theorem represents partial progress towards the construction of a model with no special Aronszajn trees on any regular cardinal greater than  $\aleph_1$  and hence the weaker goal above.

Our theorem combines two approaches to the problem, which have until now seemed incompatible. The first is a ground up approach where one constructs models where a longer and longer initial segments of the regular cardinals carry no special trees (or even have the tree property). The major advances in this approach are due to Abraham [1], Cummings and Foreman [5], Magidor and Shelah [13], Sinapova [21], Neeman [17] and the author [29] for the tree property, and Mitchell [14] and the author [28] for the nonexistence of special trees. This approach cannot continue through the first strong limit cardinal without some changes suggested by the second approach.

The second is an approach for dealing with the successors of singular strong limit cardinals. By Specker's theorem if  $\nu$  is singular strong limit and there are no special  $\nu^{++}$ -Aronszajn trees, then  $2^\nu > \nu^+$ . So the *singular cardinals hypothesis* fails at  $\nu$ . The singular cardinals hypothesis is an important property in the study of the continuum function on singular cardinals and obtaining a model where it fails requires the existence of large cardinals. Any model for the non existence of special trees above  $\aleph_1$  must be a model where GCH fails everywhere. Such a model was first constructed by Foreman and Woodin [7] using a complex Radin forcing construction.

For some time, a major problem for the second approach was whether it is consistent to have the failure of SCH at  $\nu$  and the nonexistence of special Aronszajn trees at  $\nu^+$ . This was resolved by Gitik and Sharon [9] and their result was later improved by Neeman [16] to give the tree property. Note that by our remarks above such models are required to get the nonexistence of special trees (or the tree property) at  $\nu^+$  and  $\nu^{++}$  where  $\nu$  is singular strong limit. Further advances in this area are due to Cummings and Foreman [5], the author [27], Sinapova [21, 20, 22] and the author and Sinapova [25].

The main forcing in this paper combines the two approaches outlined above. In particular it combines the ground up approach in [28] with a version of Gitik and Sharon's [9] Prikry type forcing. In the jargon, collapses which enforce the nonexistence of special trees are *interleaved* with the Prikry points. The main difficulty of the paper comes from controlling new collapses as Prikry points are added. This is typically done by leaving some gaps between the Prikry points and the associated collapses. In the current work, we have no such luxury since we wish

to control the combinatorics of every regular cardinal below the cardinal which becomes singular.

The paper is organized as follows. In Section 2, we make some preliminary definitions most of which are standard in the study of either singular cardinal combinatorics or compactness properties at double successors. In Section 3, we describe the preparation forcing and derive the measures that we need for the main forcing. In Section 4, we describe the main forcing and prove that it gives the desired cardinal structure. In Section 5, we give a schematic view of an argument for the failure of approachability at a double successor cardinal. In Section 6, we prove that the extension by the main forcing gives the desired failure of approachability. For the double successor cardinals we apply the scheme from the previous section and for the successor of each singular cardinal we apply arguments from singular cardinal combinatorics.

## 2. PRELIMINARIES

In this section we define the combinatorial notions and forcing posets which are at the heart of the paper. By a theorem of Jensen [10], the existence of a special  $\sigma^+$ -Aronszajn tree is equivalent to the existence of a *weak square sequence* at  $\sigma$ .

**Definition 2.1.** *A weak square sequence at  $\sigma$  is a sequence  $\langle \mathcal{C}_\alpha \mid \alpha < \sigma^+ \rangle$  such that*

- (1) *For all  $\alpha < \sigma^+$ ,  $1 \leq |\mathcal{C}_\alpha| \leq \sigma$ ,*
- (2) *For all  $\alpha < \sigma^+$  and all  $C \in \mathcal{C}_\alpha$ ,  $C$  is a club subset of  $\alpha$  of ordertype at most  $\sigma$  and*
- (3) *For all  $\alpha < \sigma^+$ , all  $C \in \mathcal{C}_\alpha$  and all  $\beta \in \lim(C)$ ,  $C \cap \beta \in \mathcal{C}_\beta$ .*

If there is such a sequence, then we say that *weak square holds* at  $\sigma$ .

In this paper we are interested in the weaker *approachability property* isolated by Shelah [19, 18]. For a cardinal  $\tau$  and a sequence  $\langle a_\alpha \mid \alpha < \tau \rangle$  of bounded subsets of  $\tau$ , we say that an ordinal  $\gamma < \tau$  is *approachable* with respect to  $\vec{a}$  if there is an unbounded  $A \subseteq \gamma$  such that  $\text{otp}(A) = \text{cf}(\gamma)$  and for all  $\beta < \gamma$  there is  $\alpha < \gamma$  such that  $A \cap \beta = a_\alpha$ . Using this we can define *the approachability ideal*  $I[\tau]$ .

**Definition 2.2.**  *$S \in I[\tau]$  if and only if there are a sequence  $\langle a_\alpha \mid \alpha < \tau \rangle$  and a club  $C \subseteq \tau$  such that every  $\gamma \in S \cap C$  is approachable with respect to  $\vec{a}$ .*

It is easy to see that  $I[\tau]$  contains the nonstationary ideal. We say that *the approachability property* holds at  $\tau$  if  $\tau \in I[\tau]$ . We note that weak square at  $\sigma$  implies the approachability property at  $\sigma^+$  and refer the interested reader to [3] for a proof.

In the case where  $\sigma$  is a singular cardinal, the approachability property at  $\sigma^+$  is connected with the notion of good points in Shelah's PCF theory. We give a brief description of this in the case when  $\text{cf}(\sigma) = \omega$ , since we will make use of it in the final argument below.

Suppose that  $\langle \sigma_n \mid n < \omega \rangle$  is an increasing sequence of regular cardinals cofinal in  $\sigma$ . A sequence of functions  $\langle f_\alpha \mid \alpha < \sigma^+ \rangle$  is a *scale* of length  $\sigma^+$  in  $\prod_{n < \omega} \sigma_n$  if it is increasing and cofinal in  $\prod_{n < \omega} \sigma_n$  under the eventual domination ordering. A remarkable theorem of Shelah is that there are sequences  $\langle \sigma_n \mid n < \omega \rangle$  for which scales of length  $\sigma^+$  exist.

If  $\vec{f}$  is a scale of length  $\sigma^+$  in  $\prod_{n<\omega} \sigma_n$ , then we say that  $\gamma$  is *good* for  $\vec{f}$  if there are an unbounded  $A \subseteq \gamma$  and an  $N < \omega$  such that for all  $n \geq N$ ,  $\langle f_\alpha(n) \mid \alpha \in A \rangle$  is strictly increasing. A scale  $\vec{f}$  is *good* if there is a club  $C \subseteq \sigma^+$  such that every  $\gamma \in C$  is good for  $\vec{f}$ . A scale is *bad* if it is not good. We note that approachability at  $\sigma^+$  implies that all scales of length  $\sigma^+$  are good and refer the reader to [3] for a proof.

Each of the principles described above can be thought of as an instance of incompactness. In this paper we will be interested in the negation of these properties. We summarize the implications discussed above, but in terms of the negations.

- (1) For all cardinals  $\sigma$ , the failure of approachability at  $\sigma^+$  implies the failure of weak square at  $\sigma$ .
- (2) For all singular cardinals  $\sigma$ , there is a bad scale of length  $\sigma^+$  implies the failure of approachability at  $\sigma^+$ .

In the arguments below we will either argue for the existence of a bad scale or directly for the failure of approachability. In particular we will have the failure of weak square on a long initial segment of the cardinals or equivalently the nonexistence of special Aronszajn trees on that interval.

We now define some of the forcing posets which will form the collapses between the Prikry points in our main forcing. The poset is essentially due to Mitchell [14] from his original argument for nonexistence of special  $\aleph_2$ -Aronszajn trees. We use a more flexible version of this poset as described by Neeman [17].

**Definition 2.3.** Let  $\rho < \sigma < \tau \leq \eta$  be regular cardinals and let  $\mathbb{P} = \text{Add}(\rho, \eta)$ . Let  $\mathbb{C}(\mathbb{P}, \sigma, \tau)$  be the collection of partial functions  $f$  of size less than  $\sigma$  whose domain is a set of successor ordinals contained in the interval  $(\sigma, \tau)$  such that for all  $\alpha \in \text{dom}(f)$ ,  $f(\alpha)$  is a  $\mathbb{P} \restriction \alpha$ -name for an element of  $\text{Add}(\sigma, 1)_{V[\mathbb{P} \restriction \alpha]}$ . We order the poset by  $f_1 \leq f_2$  if and only if  $\text{dom}(f_1) \supseteq \text{dom}(f_2)$  and for all  $\alpha \in \text{dom}(f_2)$ ,  $\Vdash_{\mathbb{P} \restriction \alpha} f_1(\alpha) \leq f_2(\alpha)$ .

Note that such posets are easily ‘enriched’ in the sense of Neeman to give Mitchell like collapses. The *enrichment* of  $\mathbb{C}(\mathbb{P}, \sigma, \tau)$  by  $\mathbb{P}$  is the poset defined in the generic extension by  $\mathbb{P}$  with the same underlying set as  $\mathbb{C}(\mathbb{P}, \sigma, \tau)$ , but with the order  $f_1 \leq f_2$  if and only if  $\text{dom}(f_1) \supseteq \text{dom}(f_2)$  and there is  $p$  in the generic for  $\mathbb{P}$  such that for all  $\alpha \in \text{dom}(f_2)$ ,  $p \restriction \alpha \Vdash f_1(\alpha) \leq f_2(\alpha)$ . If  $P$  is  $\mathbb{P}$ -generic we will write  $\mathbb{C}^{+P}$  for the enrichment of  $\mathbb{C}$  by  $P$ . We will drop the  $P$  and just write  $\mathbb{C}^+$  when it is clear from context which  $P$  is required. Note that the poset  $\mathbb{C}(\mathbb{P}, \sigma, \tau)$  is  $\sigma$ -closed,  $\tau$ -cc if  $\tau$  is inaccessible and collapses every regular cardinal in the interval  $(\sigma, \tau)$  to have size  $\sigma$ . Hence it makes  $\tau$  into  $\sigma^+$ .

For notational convenience we make the following definition.

**Definition 2.4.** Let  $\rho < \sigma < \tau$  be cardinals. If  $\mathbb{P} = \text{Add}(\rho, \tau)$  and  $\mathbb{C} = \mathbb{C}(\mathbb{P}, \sigma, \tau)$ , then we write  $\mathbb{M}(\rho, \sigma, \tau)$  for  $\mathbb{P} * \mathbb{C}^+$ .

We note that  $\mathbb{M}(\rho, \sigma, \tau)$  essentially gives the family of Mitchell-like posets defined in Abraham [1].

We also need two facts about term forcing. For more complete presentation of term forcing we refer the interested reader to [4]. For completeness we sketch the relevant definitions. Suppose that  $\mathbb{P}$  is a poset and  $\dot{\mathbb{Q}}$  is a  $\mathbb{P}$ -name for a poset. Let  $\mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})$  be the set of  $\mathbb{P}$ -names  $\dot{q}$  which are forced to be elements of  $\dot{\mathbb{Q}}$  with the order  $\dot{q}_1 \leq \dot{q}_2$  if  $\Vdash_{\mathbb{P}} \text{“}\dot{q}_1 \leq \dot{q}_2 \text{ in } \dot{\mathbb{Q}}\text{”}$ .

We note that  $\mathbb{P} \times \mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})$  induces a generic for  $\mathbb{P} * \dot{\mathbb{Q}}$  by taking the upwards closure in the order on  $\mathbb{P} * \dot{\mathbb{Q}}$ . This means that the extension by any two step iteration has an outer model which is the extension by a product. The  $\mathbb{C}$  posets above can be seen as a kind of term forcing and  $\mathbb{P} * \mathbb{C}^+$  can be seen as a kind of two step iteration. We will use this idea extensively in the proof.

In certain cases, the term poset can be realized as a nice poset in  $V$ .

**Fact 2.5.** *If  $\kappa^{<\kappa} = \kappa$ ,  $\lambda \geq \kappa$  and  $\mathbb{P}$  is a  $\kappa$ -cc poset, then  $\mathcal{A}(\mathbb{P}, \text{Add}^{V^{\mathbb{P}}}(\kappa, \lambda))$  is isomorphic to  $\text{Add}(\kappa, \lambda)$ .*

If  $\mathbb{A}$  is an iteration, then we can define a term ordering on  $\mathbb{A}$  by  $a \leq a'$  if the support of  $a$  contains the support of  $a'$  and for every  $\alpha$  in the support of  $a'$ , it is forced by  $\mathbb{A} \restriction \alpha$  that  $a(\alpha) \leq a'(\alpha)$ . We call this poset  $\mathcal{A}(\mathbb{A})$ . It is straightforward to see that  $\mathcal{A}(\mathbb{A})$  induces a generic for  $\mathbb{A}$ .

It is straightforward to see that the poset  $\mathcal{A}(\mathbb{A})$  has many of the same properties at the iteration. We will need the following fact.

**Fact 2.6.** *Let  $\kappa$  be a 2-Mahlo cardinal. Suppose that  $\mathbb{A}$  is an Easton support iteration of length  $\kappa$  where the set of nontrivial coordinates in the iteration is a stationary set  $S$ . If for every  $\alpha \in S$ ,  $\alpha$  is Mahlo and it is forced by  $\mathbb{A} \restriction \alpha$  that  $\mathbb{A}(\alpha)$  is  $\alpha$ -closed, then  $\mathcal{A}(\alpha)$  is  $\kappa$ -cc and preserves the Mahloness of  $\kappa$ .*

### 3. THE PREPARATION FORCING

We work in a model  $V$  and let  $\kappa$  be a supercompact cardinal. We assume that there is an increasing sequence  $\langle \kappa_i \mid i \leq \omega + 3 \rangle$  such that  $\kappa_0 = \kappa$ ,  $\kappa_\omega = \sup \kappa_i$ ,  $\kappa_{\omega+1} = \kappa_\omega^+$  and for all  $i$  (except  $\omega$ )  $\kappa_{i+1}$  is the least Mahlo cardinal above  $\kappa_i$ . For simplicity we set  $\theta = \kappa_{\omega+3}$ . For inaccessible  $\alpha$ , we define  $\langle \alpha_i \mid i \leq \omega + 3 \rangle$  and  $\alpha_{\omega+3} = \theta_\alpha$  as we did for  $\kappa$ .

We fix a supercompactness measure  $U$  on  $\mathcal{P}_\kappa(\theta)$  and let  $j : V \rightarrow M$  be the ultrapower map. It is straightforward to show that there is a set  $Z \subseteq \kappa$  such that  $\kappa \in j(Z)$  and for every  $\gamma \in Z$ ,  $\gamma$  is  $\gamma_{\omega+1}$ -supercompact and closed under the function  $\alpha \mapsto \theta_\alpha$ . In particular we have  $j(\alpha \mapsto \alpha_i)(\kappa) = \kappa_i$  for all  $i \leq \omega + 3$ .

We define an iteration with Easton support where we do nontrivial forcing at each  $\alpha \in Z$ . Suppose that we have defined  $\mathbb{A} \restriction \alpha$  for some  $\alpha \in Z$ . At stage  $\alpha$  we force with  $(\mathbb{P}_0(\alpha) * \mathbb{C}_0^+(\alpha)) \times (\mathbb{P}_1(\alpha) * \prod_{0 < i \leq \omega+1} \mathbb{C}_i^+(\alpha)) \times \text{Add}^{V[\mathbb{A} \restriction \alpha]}(\alpha_1, \theta_\alpha^+ \setminus \theta_\alpha)$  where

- (1)  $\mathbb{P}_0(\alpha) = \text{Add}(\alpha_0, \alpha_2)$  as computed in  $V[\mathbb{A} \restriction \alpha]$ ,
- (2)  $\mathbb{C}_0(\alpha) = \mathbb{C}(\mathbb{P}_0(\alpha), \alpha_1, \alpha_2)$ ,
- (3)  $\mathbb{P}_1(\alpha) = \text{Add}(\alpha_1, \theta_\alpha)$  as computed in  $V[\mathbb{A} \restriction \alpha]$  and
- (4) for all  $i$  with  $0 < i \leq \omega + 1$ ,  $\mathbb{C}_i(\alpha) = \mathbb{C}(\mathbb{P}_1(\alpha), \alpha_{i+1}, \alpha_{i+2})$ .

We take the product  $\prod_{0 < i \leq \omega+1} \mathbb{C}_i^+(\alpha)$  with full support.

For ease of notation we let  $\mathbb{A} = \mathbb{A} \restriction \kappa$ ,  $\mathbb{P} = \mathbb{P}_0(\kappa) \times \mathbb{P}_1(\kappa)$  and in the extension by  $\mathbb{P}$  we let  $\mathbb{C}^+ = \prod_{i \leq \omega+1} \mathbb{C}_i^+(\kappa)$ . We will now lift  $j$  preserving its large cardinal properties precisely with one further addition. Let  $G$  be  $\mathbb{A}$ -generic and let  $H = (H_0 * H_1) \times H_2$  be  $(\mathbb{P} * \mathbb{C}^+) \times \text{Add}(\kappa_1, \theta^+ \setminus \theta)$ -generic over  $V[G]$ .

**Lemma 3.1.** *In  $V[G * H]$ , there is a generic  $G^* * H^*$  for  $j(\mathbb{A} * (\mathbb{P} * \mathbb{C}^+) \times \text{Add}^{V[\mathbb{A}]}(\kappa_1, \theta^+ \setminus \theta))$  such that  $j$  extends to  $j : V[G * H] \rightarrow M[G^* * H^*]$  witnessing that  $\kappa$  is  $\theta$ -supercompact and for all  $\gamma < j(\kappa_1)$ , there is a function  $f : \kappa_1 \rightarrow \kappa_1$  such that  $j(f)(\sup j''\kappa_1) = \gamma$ .*

*Proof.* Much of the proof is standard, so we sketch some parts and give details where important. To construct  $G^*$  we note

- (1)  $j(\mathbb{A} \restriction \kappa) \restriction \kappa + 1 = \mathbb{A} \restriction \kappa * ((\mathbb{P} * \mathbb{C}^+) \times \text{Add}(\kappa_1, \theta^+ \setminus \theta))$  and
- (2) the poset  $j(\mathbb{A})/G * H$  is  $\theta^+$ -closed in  $V[G * H]$  and has just  $\theta^+$  antichains in  $M[G * H]$ .

Standard arguments allow us to build a generic for the tail forcing and thus form  $G^*$  which is generic for  $j(\mathbb{A})$  over  $M$  and such that  $j$  lifts to  $j : V[G] \rightarrow M[G^*]$ .

By closure properties of  $M$ ,  $j^{\text{``}}(H_0 * H_1) \in M[G^*]$  and is a directed set there of cardinality  $\theta$ . Moreover,  $j(\mathbb{P} * \mathbb{C}^+)$  is  $j(\kappa)$ -directed closed and hence we can find a master condition for  $j^{\text{``}}H_0 * H_1$ . Another routine counting of antichains allows us to build a generic  $H_0^* * H_1^*$  for  $j(\mathbb{P} * \mathbb{C}^+)$  over  $M[G^*]$  containing our master condition. At this point we can lift  $j$  to  $j : V[G * (H_0 * H_1)] \rightarrow M[G^* * (H_0^* * H_1^*)]$ .

The final difficulty is to lift to the extension by  $\text{Add}(\kappa_1, \theta^+ \setminus \theta)$ . It is here that we will control the values of  $j(f)(\sup j^{\text{``}}\kappa_1)$  for generic functions  $f$  from  $\kappa_1$  to  $\kappa_1$ . In preparation for this step we let  $\langle \eta_\gamma \mid \gamma \in \theta^+ \setminus \theta \rangle$  be an enumeration of  $j(\kappa_1)$ .

Note that  $j$  is continuous at  $\theta^+$ , so that  $j(\text{Add}(\kappa_1, \theta^+ \setminus \theta)) = \bigcup_{\gamma < \theta^+} \text{Add}(\kappa_1, j(\gamma) \setminus j(\theta))$ . Moreover if we can construct an increasing sequence of generics  $\langle H^\gamma \mid \gamma \in \theta^+ \setminus \theta \cap \text{cof}(\theta) \rangle$  with each  $H^\gamma$  generic for  $\text{Add}(\kappa_1, j(\gamma) \setminus j(\theta))$  over  $M[G^* * H_0^* * H_1^*]$ , then  $\bigcup_{\gamma < \theta^+} H^\gamma$  is generic for  $j(\text{Add}(\kappa_1, \theta^+ \setminus \theta))$  over the same model.

We construct these by induction ensuring that  $H^\gamma$  contains the condition  $p_\gamma = \bigcup_{p \in j^{\text{``}}H_2} p \restriction j(\gamma) \cup \{((j(\beta), \sup j^{\text{``}}\kappa_1), \eta_\beta) \mid \beta < \gamma\}$ . This condition (as well as genericity) is preserved by taking unions at limit stages, so we focus on the successor step. Suppose that we have constructed  $H^\gamma$  for some  $\gamma$  of cofinality  $\theta$ . Let  $\gamma^*$  be the least ordinal of cofinality  $\theta$  above  $\gamma$  it is enough to construct a generic for  $\text{Add}(\kappa_1, j(\gamma^*) \setminus j(\gamma))$  over the model  $M[G^* * (H_0^* * H_1^*) \times H^\gamma]$  containing the condition  $p_{\gamma^*} \restriction [\gamma, \gamma^*)$ . This is possible using the usual counting of antichains and the  $j(\kappa)$ -directed closure of the poset.  $\square$

For each  $n < \omega$  we can derive a supercompactness measure  $U_n$  on  $\mathcal{P}_\kappa(\kappa_n)$  from  $j$  in  $V[G * H]$ . We let  $j_n : V[G * H] \rightarrow M_n$  be the ultrapower map and let  $k_n : M_n \rightarrow M[G^* * H^*]$  be the factor map. By the way that we lifted  $j$ , for  $n \geq 1$  we have  $j(\kappa_1) + 1 \subseteq \text{ran}(k_n)$  and hence  $\text{crit}(k_n) > j(\kappa_1)$ . This property is essential in the arguments below.

We end the section with an proposition which will help us prove that the relevant cardinals are preserved by the final forcing.

**Proposition 3.2.** *For every  $\alpha \in Z \cup \{\kappa\}$  and every  $n \leq \omega + 3$  where  $n$  is not  $\omega$  or  $\omega + 1$ , there are an outer model  $W$  of  $V[G * H]$  and posets  $\bar{\mathbb{Y}}$  and  $\hat{\mathbb{Y}}$  such that*

- (1)  $W$  is an extension of  $V$  by  $\bar{\mathbb{Y}} \times \hat{\mathbb{Y}}$ ,
- (2)  $\bar{\mathbb{Y}}$  is  $\alpha_n$ -Knaster,
- (3)  $\hat{\mathbb{Y}}$  is  $\alpha_n$ -closed and
- (4) there is a generic for  $\bar{\mathbb{Y}}$  in  $V[G * H]$ .

*Proof.* Let  $\alpha$  and  $n$  be as in the proposition. We work through a few cases. First suppose that  $n = 0$ . It is straightforward to see that the iteration can be broken up as  $\mathbb{A} \restriction \alpha$  which is  $\alpha$ -cc followed by the rest of the iteration which is forced to be  $\alpha$ -closed. So we can set  $\bar{\mathbb{Y}}$  to be  $\mathbb{A} \restriction \alpha$  and  $\hat{\mathbb{Y}}$  to be the poset of  $\mathbb{A} \restriction \alpha$ -names for elements of the rest of the iteration.

Suppose that  $n = 1$ . Then there is an outer model of  $V[G * H]$  where we have a generic for  $\mathcal{A}(\mathbb{A} \restriction \alpha, \mathbb{P}_1(\alpha) \times \prod_{i \leq \omega+1} \mathbb{C}_i) \times \mathcal{A}(\mathbb{A} \restriction \alpha+1, \dot{\mathbb{A}} \restriction [\alpha+1, \kappa+1])$ . Note that this poset is  $\alpha_1$ -closed in  $V$ . So we take it to be  $\hat{\mathbb{Y}}$  and set  $\bar{\mathbb{Y}}$  to be  $\mathbb{A} \restriction \alpha * \mathbb{P}_0(\alpha)$ .

Suppose that  $n \geq 2$ . Then there is an outer model of  $V[G * H]$  where we have a generic for  $\mathcal{A}(\mathbb{A} \restriction \alpha, \prod_{n-1 \leq i \leq \omega+1} \mathbb{C}_i) \times \mathcal{A}(\mathbb{A} \restriction \alpha+1, \dot{\mathbb{A}} \restriction [\alpha+1, \kappa+1])$ . Note that this poset is  $\alpha_n$ -closed in  $V$ . So we can take it to be  $\hat{\mathbb{Y}}$  and set  $\bar{\mathbb{Y}}$  to be  $\mathbb{A} \restriction \alpha * \mathbb{P}_0(\alpha) * \mathbb{C}_0^+(\alpha) \times \mathbb{P}_1(\alpha) * \prod_{1 \leq i < n-1} \mathbb{C}_i^+$ .  $\square$

It is immediate that each such  $\alpha_n$  is preserved and for all  $n \geq 1$ ,  $\alpha_{n+1} = \alpha_n^+$  in  $V[G * H]$ . Further standard arguments using the above proposition show that  $\alpha_\omega$  and  $\alpha_{\omega+1}$  are preserved for all  $\alpha \in Z \cup \{\kappa\}$ .

#### 4. THE MAIN FORCING

In order to define a diagonal Prikry forcing we define a collection of Mitchell-like collapses which will go between the Prikry points. Let  $\mathbb{Q}(\alpha, \beta) = \mathbb{Q}^0(\alpha, \beta) \times \mathbb{Q}^1(\alpha, \beta)$  where

$$\begin{aligned} \mathbb{Q}^0(\alpha, \beta) &= \text{Add}(\alpha_{\omega+2}, \beta) * \mathbb{C}^+(\text{Add}(\alpha_{\omega+2}, \beta), \alpha_{\omega+3}, \beta) \\ \mathbb{Q}^1(\alpha, \beta) &= \text{Add}(\alpha_{\omega+3}, \beta_1) * \mathbb{C}^+(\text{Add}(\alpha_{\omega+3}, \beta_1), \beta, \beta_1) \end{aligned}$$

Extremely important to the construction is that we take all these posets as defined in  $V$ . The intention is to force  $\beta$  to become  $\alpha_{\omega+3}^+$  and  $\beta_1$  to be  $\beta^+$ . We will be sloppy and write  $\mathbb{Q}(x, y)$  for  $\mathbb{Q}(\kappa_x, \kappa_y)$ .

We are now ready to define the diagonal Prikry poset. Let  $Z_n$  be the set of  $x$  in  $\mathcal{P}_\kappa(\kappa_n)$  such that  $\kappa_x = x \cap \kappa \in Z$  and  $\text{otp}(x)$  is  $\alpha_n$  where  $\alpha = \kappa_x$ . Clearly  $Z_n \in U_n$ . For  $n < m$ ,  $x \in Z_n$  and  $y \in Z_m$ , we write  $x \prec y$  for  $x \subseteq y$  and  $|x| < \kappa_y$ .

**Definition 4.1.** Let  $\mathbb{R}$  be a poset where conditions are of the form

$$\langle q_0, x_0, q_1, x_1, \dots, q_{n-1}, x_{n-1}, f_n, F_{n+1}, F_{n+2}, \dots \rangle$$

such that

- (1)  $n \neq 0$  implies  $q_0 \in \text{Coll}(\omega, \alpha_\omega)$  where  $\alpha = \kappa_{x_0}$ .
- (2) for all  $i < n$ ,  $x_i \in Z_i$  and  $\vec{x}$  is  $\prec$ -increasing,
- (3) for all  $i \in [1, n)$ ,  $q_i \in \mathbb{Q}(x_{i-1}, x_i)$ ,
- (4) There is a sequence of measure one sets  $\langle A_i \mid i \geq n \rangle$  such that  $\text{dom}(f_n) = A_n$  and for all  $i \geq n+1$ ,  $\text{dom}(F_i) = A_i \times A_{i+1}$ ,
- (5)  $n = 0$  implies for all  $x \in A_n$ ,  $f_n(x) \in \text{Coll}(\omega, \alpha_\omega)$  where  $\alpha = \kappa_x$  and otherwise for all  $x \in A_n$ ,  $f_n(x) \in \mathbb{Q}(x_{n-1}, x)$
- (6) for all  $i \geq n+1$  and  $(x, y) \in \text{dom}(F_i)$ ,  $F_i(x, y) \in \mathbb{Q}(x, y)$ .

For a condition  $p \in \mathbb{P}$  we adorn each part of  $p$  with a superscript to indicate its connection with  $p$ . For example  $q_0^p, x_0^p$  etc. We also let  $\ell(p) = n$  denote the length of the condition  $p$ . For  $p, r \in \mathbb{P}$  we say that  $p \leq r$  if  $\ell(p) \geq \ell(r)$  and

- (1)  $\vec{x}^p$  end extends  $\vec{x}^r$  and new points come from measure one sets of  $r$ ,
- (2)  $\vec{q}^p \restriction \ell(r) \leq \vec{q}^r$
- (3)  $q_{\ell(r)}^p \leq f_{\ell(r)}^r(x_n^p)$
- (4) for all  $i \in [\ell(r) + 1, \ell(p))$ ,  $q_i^p \leq F_i^r(x_{i-1}, x_i)$
- (5) for all  $i \geq \ell(p)$ ,  $A_i^p \subseteq A_i^r$ ,
- (6) for all  $x \in A_{\ell(p)}^p$ ,  $f_{\ell(p)}^p(x) \leq F_{\ell(p)}^r(x_{\ell(p)}^p, x)$  and
- (7) for all  $i \geq \ell(p) + 1$  and all  $(x, y) \in A_i^p \times A_{i+1}^p$ ,  $F_i^p(x, y) \leq F_i^r(x, y)$ .

We fix some terminology:

- (1) We call the lower part or stem of a condition any sequence of the form  $\langle q_0, x_0, q_1, \dots, x_{n-1} \rangle$ . Note that we have not included  $f_n$ .
- (2) If  $r$  is a condition, then we write  $\text{stem}(r)$  to denote the stem of  $r$ .
- (3) If  $s = \langle q_0, x_0, q_1, \dots, x_{n-1} \rangle$  is a stem, then we let  $\text{top}(s) = x_{n-1}$ .
- (4) We call the one variable functions  $f_n^p$  the  $f$ -part of  $p$ .
- (5) We call the sequence of two variable functions  $\vec{F}^p$  the upper part or constraint of  $p$ .

For each  $i \geq 1$ , the class of a function  $F_i$  as above modulo  $U_i \times U_{i+1}$  is a member of  $\mathbb{Q}_i = \mathbb{Q}(\kappa, j_i(\kappa))$  as computed in the (external) ultrapower of  $V$  by  $U_i \times U_{i+1}$ . This forcing is a product of Mitchell-like collapses below  $j_i(\kappa_1)$  which is the least (formerly) Mahlo cardinal above  $j_i(\kappa)$ .

We want to show that the appropriate factor map which pushes  $\mathbb{Q}_i$  into  $\text{Ult}(V[G * H], U \times U)$  has critical point above  $j(\kappa_1)$ . It follows that  $\mathbb{Q}_i$  is equal to  $\mathbb{Q}(\kappa, j_i(\kappa))$  as computed in  $\text{Ult}(V, U \times U)$ . However, this second iterated ultrapower is highly closed inside  $M$  and hence  $\mathbb{Q}_i$  is  $\mathbb{Q}(\kappa, j_i(\kappa))$  as computed in  $M$ .

We fix some notation. For  $i \geq 1$ ,  $M_i^{i+1} = \text{Ult}(V[G * H], U_i \times U_{i+1})$  and  $j_i(M[G * H * H^*]) = \text{Ult}(M_i, j_i(U))$ . By standard arguments there are elementary embeddings,  $k : M_i^{i+1} \rightarrow j_i(M[G * H * H^*])$  and  $\hat{k} : j_i(M[G * H * H^*]) \rightarrow \text{Ult}(V[G * H], U \times U)$ . Note that  $k$  is given by  $j_i(k_{i+1})$  and hence has critical point above  $j_i(j(\kappa_1))$ . Further note that  $\hat{k}$  is given by the restriction of  $k_i$  to the domain of  $\hat{k}$  and hence has critical point above  $j(\kappa_1)$ . So it follows that for all  $i \geq 1$ ,  $\mathbb{Q}_i$  is  $\mathbb{Q}(\kappa, j_i(\kappa))$  as computed in  $M$ . Of course using the high critical point of  $k_i$ ,  $j_i(\kappa) = j(\kappa)$ . So we have shown that each  $\mathbb{Q}_i$  for  $i \geq 1$  is in fact the same forcing and we drop the dependence on  $i$  and just write  $\mathbb{Q}$ .

**Claim 4.2.** *The full support power  $\mathbb{Q}^\omega$  is  $< \kappa_{\omega+2}$ -distributive in  $V[G * H]$ .*

*Proof.*  $\mathbb{Q}$  is defined in  $M$  (hence  $V$ ) and is  $\kappa_{\omega+2}$ -closed in  $V$  by the closure of  $M$ . Hence  $\mathbb{Q}^\omega$  is  $\kappa_{\omega+2}$  closed in  $V$ . Let  $W$ ,  $\bar{\mathbb{Y}}$  and  $\hat{\mathbb{Y}}$  be the outer model and posets from Proposition 3.2 applied to  $\kappa_{\omega+2}$ . By Easton's lemma, every  $< \kappa_{\omega+2}$ -sequence from  $W^{\mathbb{Q}^\omega}$  is in  $V^{\bar{\mathbb{Y}}}$ , but  $V[G * H]$  contains a generic for  $\bar{\mathbb{Y}}$ . So we are done.  $\square$

A similar argument establishes the following claim.

**Claim 4.3.** *For all  $\alpha < \beta \leq \kappa$ ,  $\mathbb{Q}(\alpha, \beta)$  is  $< \alpha_{\omega+2}$ -distributive in  $V[G * H]^{\mathbb{Q}^\omega}$ .*

**Remark 4.4.** *It is clear from the proof above that the conclusion of the previous claim holds in any forcing extension by a poset from  $V$  of size less than  $\alpha_{\omega+1}$ . The small poset can be included in  $\bar{\mathbb{Y}}$ .*

We pause here to prove that the posets  $\mathbb{Q}(\alpha, \beta)$  have the desired effect on the cardinals between  $\alpha$  and  $\beta$ .

**Proposition 4.5.** *In any extension of  $V[G * H]$  by a poset of size less than  $\alpha_{\omega+1}$  from  $V$ ,  $\mathbb{Q}(\alpha, \beta)$  is  $\beta_1$ -cc, preserves the cardinals  $\alpha_{\omega+3}$  and  $\beta$  and forces  $\beta = \alpha_{\omega+3}^+$  and  $\beta_1 = \beta^+$ .*

*Proof.* Let  $\mathbb{Y}$  be a poset of size less than  $\alpha_{\omega+1}$  in  $V$ . First we consider the outer model  $W$  and posets  $\bar{\mathbb{Y}}$  and  $\hat{\mathbb{Y}}$  from Proposition 3.2 applied to  $\beta_1$ . In  $V$ , the product  $\mathbb{Y} \times \bar{\mathbb{Y}} \times \mathbb{Q}(\alpha, \beta)$  is  $\beta_1$ -Knaster. By Easton's lemma, it follows that  $\mathbb{Q}(\alpha, \beta)$  is  $\beta_1$ -cc in  $W^{\mathbb{Y}}$  and hence in  $V[G * H]^{\mathbb{Y}}$ .



Second we consider the outer model  $W$  and posets  $\bar{\mathbb{Y}}$  and  $\hat{\mathbb{Y}}$  from Proposition 3.2 applied to  $\beta$ . Using the definition of the  $\mathbb{C}$  posets, there is an outer model of  $V^{\mathbb{Q}(\alpha, \beta)}$  which is an extension by the partial order  $\mathbb{Q}^0(\alpha, \beta) \times \text{Add}(\alpha_{\omega+3}, \beta_1) \times \mathbb{C}(\text{Add}(\alpha_{\omega+3}, \beta_1), \beta, \beta_1)$ . Hence  $W^{\mathbb{Q}(\alpha, \beta)}$  is contained in a generic extension by the product of  $\mathbb{Y} \times \bar{\mathbb{Y}} \times \mathbb{Q}^0(\alpha, \beta) \times \text{Add}(\alpha_{\omega+3}, \beta_1)$  and  $\hat{\mathbb{Y}} \times \mathbb{C}(\text{Add}(\alpha_{\omega+3}, \beta_1), \beta, \beta_1)$ . The first piece is  $\beta$ -cc and the second is  $\beta$ -closed. Hence  $\beta$  is preserved in  $V[G * H]^{\mathbb{Y} \times \mathbb{Q}(\alpha, \beta)}$ .

The argument that  $\alpha_{\omega+3}$  is preserved is similar to the argument that  $\beta$  is preserved, but we split up  $\mathbb{Q}^0(\alpha, \beta)$  instead of  $\mathbb{Q}^1(\alpha, \beta)$  and incorporate  $\mathbb{Q}^1(\alpha, \beta)$  into the closed part.

The argument that cardinals in the intervals  $(\alpha_{\omega+3}, \beta)$  and  $(\beta, \beta_1)$  are collapsed is standard for Mitchell type posets.  $\square$

**Claim 4.6.** *Forcing with  $\mathbb{Q}^\omega$  over  $V[G * H]$  preserves cardinals up to  $\kappa_{\omega+3}$ .*

*Proof.* By Claim 4.2, it is enough to show that  $\kappa_{\omega+3}$  is preserved. Recall that  $\mathbb{Q}$  is computed in an ultrapower of  $V$  which is closed under  $\theta = \kappa_{\omega+3}$ -sequences. In particular, it can be written as the projection of a product  $\text{Add}(\kappa_{\omega+2}, j(\kappa))$  which is  $\kappa_{\omega+3}$ -cc in  $V$  and  $\mathbb{C}(\text{Add}(\kappa_{\omega+2}, j(\kappa)), \kappa_{\omega+3}, j(\kappa)) \times \mathbb{Q}^1(\kappa, j(\kappa))$  which is  $\kappa_{\omega+3}$ -closed in  $V$ . Since the iteration to add  $G * H$  is  $\kappa_{\omega+3}$ -cc, we have that  $\kappa_{\omega+3}$  is preserved when we force with  $\mathbb{Q}^\omega$ .  $\square$

We are now ready to prove the Prikry lemma. The main elements come from a combination of [2] and [23]. Suppose that we force with  $\mathbb{Q}^\omega$  to obtain  $\vec{K} = \langle K_n \mid n \geq 1 \rangle$ . We let  $\bar{\mathbb{R}}$  be the set of conditions  $r$  such that  $\langle [F_n^r] \mid n \geq \max(\ell(p), 1) \rangle \in \prod_{n \geq \max(\ell(p), 1)} K_n$  ordered as a suborder of  $\mathbb{R}$ . The following claims are straightforward.

**Claim 4.7.** *In  $V[G * H]^{\mathbb{Q}^\omega}$ ,  $\bar{\mathbb{R}}$  is  $\kappa_\omega$ -centered below any condition of length at least one.*

Note that in  $\bar{\mathbb{R}}$ , conditions of length at least one with the same stem and (equivalence class of)  $f$ -part are compatible and there are at most  $\kappa_\omega$  such pairs.

**Claim 4.8.**  *$\mathbb{Q}^\omega * \bar{\mathbb{R}}$  projects to  $\mathbb{R}$ .*

This claim allows us to prove the Prikry lemma for  $\bar{\mathbb{R}}$  in place of  $\mathbb{R}$ .

**Lemma 4.9.** *Work in  $V[G * H][\vec{K}]$ . Let  $r \in \bar{\mathbb{R}}$  be a condition of length at least 1 and  $\varphi$  be a statement in the forcing language. There is an  $r^* \leq^* r$  which decides  $\varphi$ .*

It follows immediately that  $\bar{\mathbb{R}}$  itself has the Prikry property, since the projection from the previous claim fixes the length of a condition. We break the proof into many rounds.

**Claim 4.10.** *There is an  $r_0 \leq^* r$  such that for all  $p \leq r_0$  if  $p$  is at least a one point extension of  $r_0$  and it decides  $\varphi$ , then there is an upper part  $\vec{F}$  such that  $\text{stem}(p) \cap F_{\ell(p)}^{r_0}(x_{\ell(p)-1}^p) \cap \vec{F}$  decides  $\varphi$ .*

*Proof.* We go by induction on the length of extensions of  $r$ . Let  $r^{\ell(r)} = r$ . Assume that we have constructed  $r^n$  for some  $n < \omega$ . Work in the ultrapower  $M_n$  and consider conditions of length  $n$  of the form

$$s \cap (q, j_n \text{ `` } \kappa_n) \cap (f^+, \vec{F}^+) \leq \text{stem}(r^n) \cap (j_n(f^{r^n}), j_n(\vec{F}^{r^n}))$$

Note that  $f^+ \leq j_n(F_n^{r^n})(j_n \text{``}\kappa_n)$  in particular there is a function  $F^*$  such that  $f^+ = j(F^*)(j_n \text{``}\kappa_n)$ . For each  $s$  and  $q$ , the set  $D_{s,q}$  defined as

$$\{f^+ \mid \text{if } \exists \vec{F}^+, \exists f^{++} \leq f^+ \text{ such that } s \smallfrown (q, j_n \text{``}\kappa_n) \smallfrown (f^{++}, \vec{F}^+) \text{ decides } \varphi \\ \text{then } \exists \vec{F}^{++} s \smallfrown (q, j_n \text{``}\kappa_n) \smallfrown (f^+, \vec{F}^{++}) \text{ decides } \varphi\}$$

is dense open in  $\mathbb{Q}_n$ , which is just  $\mathbb{Q}$ . Moreover it can be defined in the model  $V[G * H][\langle K_i \mid i > n \rangle]$  where  $\mathbb{Q}$  is  $< \kappa_{\omega+2}$ -distributive. So the set  $D = \bigcap_{s,q} D_{s,q}$  is dense open in  $\mathbb{Q}$ . Since  $D$  can be defined in  $V[G * H][\langle K_i \mid i > n \rangle]$  and by the product lemma  $K_n$  is generic for  $\mathbb{Q}$  over this model, we can take a function  $F_n$  so that  $[F_n]_{U_n \times U_{n+1}} \in K_n$  and  $j_n(F_n)(j_n \text{``}\kappa_n) \in D$ . We can assume that  $F_n \leq F_n^{r^n}$  on a  $U_n \times U_{n+1}$  large set.

By Los' theorem the set  $A_n$  given by

$$\{x \mid \forall s \text{ if } \text{top}(s) = x, \exists \vec{F}^+ \exists f^{++} \leq F_n(x), s \smallfrown (f^{++}, \vec{F}^+) \text{ decides } \varphi \\ \text{then } \exists \vec{F}^{++}, s \smallfrown (F_n(x), \vec{F}^{++}) \text{ decides } \varphi\}$$

is  $U_n$  measure one. We define  $r^{n+1}$  by refining  $F_n^{r^n}$  to  $F_n \restriction A$  and leaving the rest of  $r^n$  unchanged.

We let  $r_0$  be a lower bound for  $\langle r^n \mid n \geq \ell(r) \rangle$ . It is straightforward to check that  $r_0$  satisfies the conditions of the claim.  $\square$

**Claim 4.11.** *There is an  $r_1 \leq^* r_0$  such that if  $p \leq r_1$  is at least a one point extension and it decides  $\varphi$ , then  $\text{stem}(p) \smallfrown F_{\ell(p)-1}^{r_1}(x_{\ell(p)-1}^p) \smallfrown \vec{F}^{r_1}$  decides  $\varphi$ .*

*Proof.* We collect witnesses to the previous claim. Let  $s$  be a stem which is at least a one point extension of  $r_0$ . If possible we select an upper part  $\vec{G}^s$  witnessing that the condition with stem  $s$  from the previous claim decides  $\varphi$ . Using the distributivity of  $\mathbb{Q}^\omega$  (in particular that the sequence of generics  $\vec{K}$  is closed), for each  $n \geq \ell(r_0) + 1$  we can find  $\vec{G}^n$  such that for all  $k \geq n$  and all  $s$  of length  $n$ ,  $[G_k^n] \leq [G_k^s]$ . It is straight forward to find a sequence  $\langle G_k \mid k \geq \ell(r_0) + 1 \rangle$  such that for all  $n \geq \ell(r_0) + 1$ ,  $[G_k] \leq [G_k^n]$ .

For each stem  $s$  there is a sequence of measure one sets  $\vec{A}^s$  on which  $\vec{G} \restriction [\ell(s), \omega)$  is below  $\vec{G}^s$ . We can assume that the sequence  $\vec{A}^s$  is contained (pointwise) in the sets which form the domains of  $\vec{G}^s$ . By a standard argument there is a sequence of measure one sets  $\langle A_n \mid n \geq \ell(r_0) + 1 \rangle$  such that for all  $x \in A_n$  if  $s$  is a stem with  $s \prec x$ , then  $x \in A_n^s$ .

We obtain  $r_1$  by for all  $n \geq \ell(r_0) + 1$  restricting  $F_n^{r_0}$  to  $G_n \restriction A_n$ . It is straightforward to check that this condition satisfies the claim.  $\square$

**Claim 4.12.** *There is a condition  $r_2 \leq^* r_1$  such that if  $p \leq r_2$  is at least a two point extension and it decides  $\varphi$ , then*

$$\text{stem}^-(p) \smallfrown (F_{\ell(p)-2}^{r_2}(x_{\ell(p)-2}^p, x_{\ell(p)-1}^p), x_{\ell(p)-1}) \smallfrown (F_{\ell(p)-1}^{r_2}(x_{\ell(p)-1}^q), \vec{F}^{r_2})$$

*decides  $\varphi$ .*

*Proof.* We work by induction on the lengths of possible extensions of  $r_1$  of at least two points. Let  $r_1 = r^{\ell(r)}$  and assume that we have constructed  $r^n$  for some  $n$ . We work in  $M_n^{n+1}$  and consider conditions of the form:

$$s^- \smallfrown (j_n^{n+1} \text{``}\kappa_n, q, k_n^{n+1} \text{``}(j_n(\kappa_{n+1}))) \smallfrown (j_n^{n+1}(f^{r^n}), j_n^{n+1}(\vec{F}^{r^n}))$$

Note that  $q \in \mathbb{Q}$ . We denote this condition  $r_{s,q}$ . For each  $s$  of length  $n$ , the set  $D_s$

$$\{q \mid r_{s,q} \text{ decides } \varphi \text{ or for no extension } q' \text{ of } q \text{ does } r_{s,q'} \text{ decide } \varphi\}$$

is dense in  $\mathbb{Q}$  and defined in  $V[G * H][\langle K_m \mid m > n \rangle]$ . Using the distributivity of  $\mathbb{Q}$  in this model, the set  $D = \bigcap_{\ell(s)=n} D_s$  is dense in  $\mathbb{Q}$ . So we can find a function  $F_n$  such that  $[F_n] \in D$  with  $F_n \leq F_n^{r^n}$ .

By Los' theorem the set

$$\{(x, y) \mid s^- \frown (x, F_n(x, y), y) \frown (f^{r^n}, \vec{F}^{r^n}) \text{ decides } \varphi \\ \text{or there is no extension } q \leq F_n(x, y) \text{ which decides } \varphi\}$$

is measure one for  $U_n \times U_{n+1}$ . We fix measure one sets  $A_n^s, A_{n+1}^s$  so that every  $(x, y) \in A_n^s \times A_{n+1}^s$  with  $x \prec y$  is in the above set. Again by a standard construction we can find  $A_n \times A_{n+1}$  such that for all pairs  $(x, y) \in A_n \times A_{n+1}$  if  $s \prec x \prec y$ , then  $(x, y) \in A_n^s \times A_{n+1}^s$ . We refine  $r^n$  to  $r^{n+1}$  by replacing  $F_n^{r^n}$  with  $F_n \upharpoonright A_n \times A_{n+1}$ .

At the end of the construction we let  $r_2$  be a lower bound for  $r^n$  for  $n \geq \ell(r_1)$ . It is straightforward to check that  $r_2$  has the desired property.  $\square$

Given a condition  $p$ , a stem  $s$  extending  $p$  of length  $n$  and  $x \prec y$  for  $(x, y) \in A_n^p \times A_{n+1}^p$ , we denote the condition obtained by extending the stem of  $p$  to  $s$  and then further adjoining  $x$  and  $y$  by  $p(s, x, y)$ . For each stem  $s$  of length  $n$ , we find measure one sets  $B_n^s, B_{n+1}^s$  such that  $r_2(s, x, y)$  either forces  $\varphi$  for all  $(x, y) \in B_n^s \times B_{n+1}^s$  or forces  $\neg\varphi$  for all  $(x, y) \in B_n^s \times B_{n+1}^s$  or does not decide  $\varphi$  for all  $(x, y) \in B_n^s \times B_{n+1}^s$ .

Again by a standard argument we can restrict the measure one sets of  $r_2$  to find  $r_3$  such that for all stems  $s$  if  $p$  is below  $r_3$  with stem  $s$ , then for all  $x, y$  with  $s \prec x \prec y$  and  $(x, y) \in A_n^{r_3} \times A_{n+1}^{r_3}$ ,  $(x, y) \in B_n^s \times B_{n+1}^s$ .

Let  $p^* \leq r_3$  be an extension of minimal length deciding  $\varphi$ . We assume without loss of generality that it forces  $\varphi$ .

**Claim 4.13.**  $\ell(p^*) \leq \ell(r_3) + 1$

*Proof.* Suppose not. Then  $p^*$  is at least a two point extension of  $r_3$ . Let  $s$  be stem( $p^*$ )  $\upharpoonright \ell(p) - 2$  and let  $x = x_{\ell(p^*)-2}^{p^*}$  and  $y = x_{\ell(p^*)-1}^{p^*}$ . From our previous claims,  $r_2(s, x, y)$  forces  $\varphi$ . Hence  $(x, y)$  must come from  $B_n^s \times B_{n+1}^s$  and for every other pair  $(x', y') \in B_n^s \times B_{n+1}^s$   $r_2(s, x', y')$  forces  $\varphi$ . This contradicts the minimality of the length of  $p^*$ , since the condition  $r_2(s, x)$  (with the obvious interpretation) must force  $\varphi$ .  $\square$

**Claim 4.14.** *There is  $r_4 \leq^* r_3$  which decides  $\varphi$ .*

*Proof.* For each  $x \in A_{\ell(r_3)}^{r_3}$ , we record a condition  $q_x \leq f_{\ell(r_3)}^{r_3}(x)$  such that the condition

$$\text{stem}(r_3) \frown (q_x, x) \frown (F_{\ell(r_3)}^{r_3}(x), \vec{F}^{r_3})$$

either decides  $\varphi$  or no extension of  $q_x$  in the above condition decides  $\varphi$ . Find a measure one set  $A$  of  $x$  which all give the same decision. Let  $r_4$  be obtained from  $r_3$  by restricting  $f_n^{r_3}$  to the function  $x \mapsto q_x$  on  $A$ . Now by the previous lemma there is a one point extension of  $r_4$  which decides  $\varphi$ , hence all one point extensions of  $r_4$  decide  $\varphi$  in the same way. Hence  $r_4$  decides  $\varphi$ .  $\square$

This finishes the proof of the Priky lemma.

**Corollary 4.15.** *In the extension by  $\mathbb{R}$ ,  $\kappa = \aleph_{\omega^2}$  and if  $\kappa_n = x_n \cap \kappa$ , then  $\kappa_{n,i}$  is preserved for all  $i \leq \omega + 3$ .*

*Proof.* By Remark 4.4 and Proposition 4.5, it is enough to show that if  $\dot{X}$  is an  $\mathbb{R}$ -name for a subset of some  $\mu < \kappa$  and  $r$  is a condition with  $\mu < \lambda = \kappa \cap \text{top}(\text{stem}(r))$ , then  $r$  forces  $\dot{X}$  is in the extension by  $\prod_{i < \ell(r)} \mathbb{Q}(x_{i-1}^r, x_i^r)$ .

By the Prikry Lemma,  $\dot{X}$  is in any extension by  $(\mathbb{R}, \leq^*)$  below the condition  $r$ . This forcing decomposes as the product of  $\prod_{i < \ell(r)} \mathbb{Q}(x_{i-1}^r, x_i^r)$  and the forcing of upper parts ordered by direct extension. By an argument similar to the one from Claim 4.3, the forcing of upper parts ordered by direct extension is  $< \kappa \cap x_{\ell(r)}^r$ -distributive in the extension by  $\prod_{i < \ell(r)} \mathbb{Q}(x_{i-1}^r, x_i^r)$ . So we have the desired conclusion.  $\square$

Recall that  $\mathbb{Q}^\omega$  is the full support product. Clearly both  $\mathbb{R}$  and  $\mathbb{Q}^\omega$  project to  $\mathbb{Q}^\omega/\text{fin}$ . Let  $I$  be the  $\mathbb{Q}^\omega/\text{fin}$ -generic induced by  $\vec{K}$ .

**Claim 4.16.**  $\mathbb{R}/I$  has the  $\kappa_{\omega+1}$ -Knaster property.

*Proof.* Work in  $V[G * H][I]$  and let  $\langle r_\alpha \mid \alpha < \kappa_{\omega+1} \rangle$  be a sequence of elements of  $\mathbb{R}/I$ . We can assume that each  $r_\alpha$  has some fixed length  $l$ .

Let  $\langle [F_i] \mid i < \omega \rangle/\text{fin}$  be a condition in  $I$  forcing this. By the distributivity of  $\mathbb{Q}^\omega/\text{fin}$  we can assume that the  $F_i$  have the property that for each  $\alpha$  there is an  $n_\alpha$  such that for all  $i \geq n_\alpha$ ,  $[F_i] \leq [F_i^{r_\alpha}]$ . By passing to an unbounded subset of the  $r_\alpha$ , we can assume there is  $n^*$  such that  $n^* = n_\alpha$  for all  $\alpha < \kappa_{\omega+1}$ . Further, extending each  $r_\alpha$  if necessary we can assume that  $l = \ell(r_\alpha) \geq n^*$ . By passing to a further unbounded subset, we can assume that for all  $i < l$ ,  $x_i^{r_\alpha} = x_i^{r_\beta}$  for all  $\alpha$  and  $\beta$ .

Now for each  $\alpha$ ,  $r_\alpha \restriction l+1$  essentially comes from the poset  $\prod_{i < l} \mathbb{Q}(x_{i-1}^r, x_i^r) \times \mathbb{Q}(x_{l-1}, j_l \text{“}\kappa_l)$  where the latter forcing is computed in  $M_l$ . This forcing has cardinality less than  $\kappa_\omega$  and hence we can find an unbounded set of  $\alpha$  on which any  $r_\alpha$  and  $r_\beta$  are compatible.  $\square$

**Corollary 4.17.**  $\mathbb{R}$  preserves the cardinals  $\kappa_{\omega+1}$ ,  $\kappa_{\omega+2}$  and  $\kappa_{\omega+3}$ .

By Claim 4.6 all three cardinals are preserved in  $V[I]$  which is an inner model of an extension by  $\mathbb{Q}^\omega$  and by the previous Claim they are preserved in the extension by  $\mathbb{R}$ .

## 5. A SCHEMATIC VIEW OF ARGUMENTS FOR THE FAILURE OF APPROACHABILITY AT DOUBLE SUCCESSORS

In this section we give an abstract overview of arguments for the failure the approachability property at double successor cardinals. Before we begin, note that none of the cardinals and posets in this section bear any relation to those defined elsewhere in the paper.

We begin remarking that the approachability property is upwards absolute to models with the same cardinals. So to prove that it fails in some model, it is enough to show that it fails in an outer model with the same cardinals. We formalize this in the following remark.

**Remark 5.1.** Suppose that  $W \subseteq W'$  are models of set theory and  $\lambda$  is a regular cardinal in  $W'$ . If  $\lambda \notin I[\lambda]$  in  $W'$ , then  $\lambda \notin I[\lambda]$  in  $W$ .

We will also use a theorem of Gitik and Krueger [8] which allows us to preserve the failure of approachability in some outer models.

**Theorem 5.2.** *Suppose that  $\lambda = \mu^{++}$  and  $\mathbb{P}$  is  $\mu$ -centered. If  $\lambda \notin I[\lambda]$  in  $V$ , then it is forced by  $\mathbb{P}$  that  $\lambda \notin I[\lambda]$ .*

The bulk of this section is devoted to giving an abstract view of the failure of approachability in the extension by the Mitchell posets as described in Section 2. In particular we need to show that these posets have approximation properties.

**Definition 5.3** (Hamkins). *Let  $\mathbb{P}$  be a poset and  $\kappa$  be a cardinal. We say that  $\mathbb{P}$  has the  $\kappa$ -approximation property if for every ordinal  $\mu$  and every name  $\dot{x}$  for a subset of  $\mu$ , if for all  $z \in \mathcal{P}_\kappa(\mu)$   $\Vdash_{\mathbb{P}} \dot{x} \cap z \in V$ , then  $\Vdash_{\mathbb{P}} \dot{x} \in V$ .*

We consider the following general situation. Let  $\rho < \sigma < \tau$  be regular cardinals with  $\tau$  Mahlo and let  $\mathbb{M} = \mathbb{M}(\rho, \sigma, \tau)$  as defined in Section 2. Let  $\mathbb{X}$  be a poset such that for all  $\alpha \leq \tau$ ,  $\mathbb{X}$  is  $\alpha$ -cc in the extension by  $\mathbb{M} \restriction \alpha$ . (Here  $\mathbb{M} \restriction \tau = \mathbb{M}$ .) Suppose that  $\langle \dot{a}_\gamma \mid \gamma < \tau \rangle$  and  $\dot{C}$  are  $\mathbb{M} \times \mathbb{X}$ -names for witnesses that  $\tau \in I[\tau]$ .

We assume that there are a club  $D$  and subforcings  $\mathbb{X} \restriction \alpha$  of  $\mathbb{X}$  for  $\alpha \in D \cup \{\tau\}$  such that for all  $\alpha \in D \cup \{\tau\}$ ,  $\langle \dot{a}_\gamma \mid \gamma < \alpha \rangle$  is in the extension by  $\mathbb{M} \restriction \alpha \times \mathbb{X} \restriction \alpha$ . By the  $\tau$ -cc of  $\mathbb{M} \times \mathbb{X}$  we can assume that  $\dot{C}$  is in  $V$ , so we rename it  $C$ . Since  $\tau$  is Mahlo, there is an inaccessible  $\alpha$  in  $D \cap C$ .

It follows that  $\alpha = \sigma^+$  in the extension by  $\mathbb{M} \restriction \alpha \times \mathbb{X} \restriction \alpha$  and  $\alpha \in I[\alpha]$  as witnessed by  $\langle \dot{a}_\gamma \mid \gamma < \alpha \rangle$  and  $C \cap \alpha$ . Since  $\alpha$  is approachable in the extension by  $\mathbb{M} \times \mathbb{X}$ , there is an  $\mathbb{M}/\mathbb{M} \restriction \alpha \times \mathbb{X}/\mathbb{X} \restriction \alpha$ -name  $\dot{A}$  for a subset of  $\alpha$  of ordertype  $\sigma$  such that for all  $\delta < \alpha$ ,  $\dot{A} \cap \delta = a_\gamma$  for some  $\gamma < \alpha$ .

Since forcing with  $\mathbb{X}$  over the extension by  $\mathbb{M} \restriction \alpha$  preserves  $\alpha$ , for the failure of approachability at  $\tau$  it is enough to show that  $\mathbb{M}/\mathbb{M} \restriction \alpha$  has the  $\lambda$ -approximation property in the extension by  $\mathbb{X} \times \mathbb{M} \restriction \alpha$  for some  $\lambda \leq \sigma$ .

In the arguments for the failure of approachability below the choice of  $\mathbb{X}$  will vary, but we can prove a single lemma which captures most of the different choices. We say that a poset  $\mathbb{X}$  preserves the  $\lambda$ -cc of a poset  $\mathbb{P}$  if  $\mathbb{P}$  is  $\lambda$ -cc in the extension by  $\mathbb{X}$ .

**Lemma 5.4.** *Let  $\rho < \sigma < \tau$  be cardinals and let  $\mathbb{M} = \mathbb{M}(\rho, \sigma, \tau)$ . Let  $\lambda \leq \sigma$  be a cardinal and let  $\mathbb{X}$  be a poset such that*

- (1) *for all  $\alpha \leq \tau$ ,  $\mathbb{X}$  is  $\alpha$ -cc in the extension by  $\mathbb{M} \restriction \alpha$  and*
- (2)  *$\mathbb{X} \simeq \bar{\mathbb{X}} \times \hat{\mathbb{X}}$  where  $\bar{\mathbb{X}}^2$  is  $\lambda$ -cc and  $\hat{\mathbb{X}}$  is  $< \lambda$ -distributive and preserves the  $\lambda$ -cc of  $\text{Add}(\rho, \tau)$  and  $\bar{\mathbb{X}}$ .*

*Then in the extension by  $\mathbb{M} \restriction \alpha \times \mathbb{X}$ ,  $\mathbb{M}/\mathbb{M} \restriction \alpha$  has the  $\lambda$ -approximation property.*

The lemma is immediate from the proof of Lemma 4.1 of [25]. As stated the lemma involves a Prikry type forcing, which we can take to be trivial. We apply that lemma in the model  $V[\hat{\mathbb{X}}]$  to the forcing  $\mathbb{M} \times \bar{\mathbb{X}}$  in place of the forcing  $\mathbb{M} \times \mathbb{A}$ . In [25],  $\mathbb{A}$  is a forcing to add some Cohen subsets but we only used its chain condition in the proof. Note further that the forcing  $\hat{\mathbb{X}}$  preserves the chain condition and closure of the posets that are needed in the proof.

In the arguments below there is one double successor cardinal where the above lemma is not sufficient to give the failure of approachability. So we prove the following strengthening.

**Lemma 5.5.** *Let  $\rho < \sigma < \tau$  be cardinals where  $\sigma = \nu^+$  and  $\nu$  is a singular cardinal of cofinality  $\omega$ . Let  $\langle \nu_n \mid n < \omega \rangle$  be an increasing sequence of regular cardinals which is cofinal in  $\nu$  with  $\nu_0 > 2^{<\rho}$ . Let  $\mathbb{M} = \mathbb{M}(\rho, \sigma, \tau)$ , let  $\mathbb{E}$  be a poset of cardinality at most  $\nu$  and let  $\mathbb{X}$  be a  $\nu_0$ -closed poset such that*

- (1) for all  $\alpha \leq \tau$ ,  $\mathbb{X}$  is  $\alpha$ -cc in the extension by  $\mathbb{M} \restriction \alpha$  and
- (2) for all  $n < \omega$ ,  $\mathbb{X} \simeq \mathbb{X}_n \times \mathbb{X}^n$  where  $\mathbb{X}_n$  is  $\nu_n$ -cc and  $\mathbb{X}^n$  is  $\nu_n$ -closed.

Then in the extension by  $\mathbb{M} \restriction \alpha \times \mathbb{X} \times \mathbb{E}$ ,  $\mathbb{M}/\mathbb{M} \restriction \alpha$  has the  $\sigma$ -approximation property.

*Proof.* Working in  $V[\mathbb{M} \restriction \alpha \times \mathbb{X} \times \mathbb{E}]$ , let  $\dot{d}$  be an  $\mathbb{M}/\mathbb{M} \restriction \alpha$ -name for a subset of some ordinal  $\mu$  and assume that for all  $z \in \mathcal{P}_\sigma(\mu)$ , it is forced that  $z \cap \dot{d}$  is in the ground model.

**Claim 5.6.** *In  $V[\mathbb{M} \restriction \alpha \times \mathbb{X} \times \mathbb{E}]$ , there is a condition  $(p, f) \in \mathbb{M}/\mathbb{M} \restriction \alpha$  such that for all  $z \in \mathcal{P}_\sigma(\mu)$  if  $(p', f') \leq (p, f)$  decides  $\dot{d} \cap z$ , then so does  $(p, f')$ .*

Note that  $\mathbb{M}$  is  $\mathbb{P} * \mathbb{C}^+$  where  $\mathbb{P}$  is  $\text{Add}(\rho, \tau)$  and  $\mathbb{C} = \mathbb{C}(\mathbb{P}, \sigma, \tau)$ . We also have that for each inaccessible  $\alpha$  in the interval  $(\sigma, \tau)$  in the extension by  $\mathbb{M} \restriction \alpha$ ,  $\mathbb{M}/\mathbb{M} \restriction \alpha$  is subsumed by the product  $\mathbb{P}/\mathbb{P} \restriction \alpha \times \mathbb{C}^{+P \restriction \alpha} \restriction [\alpha, \tau)$ . We call this latter poset *the poset of  $f$ -parts under the term ordering* and note that it is  $\sigma$ -closed in the extension by  $\mathbb{M} \restriction \alpha$ . For a proof, we refer the reader to Claim 4.15 of [17].

*Proof of Claim 5.6.* We suppose otherwise. In particular, for every condition  $(p, f)$  there are a  $z \in \mathcal{P}_\sigma(\lambda)$  and a condition  $(p', f') \leq (p, f)$  such that  $(p', f')$  decides  $\dot{d} \cap z$  but  $(p, f')$  does not. In particular there is an extension of  $(p, f')$  which decides a value different from the one given by  $(p', f')$ .

Collecting this information we have that for any  $(p, f)$  there are extensions  $p_0, p_1 \leq p$  and  $f^* \leq f$  in the term ordering, such that  $(p_0, f^*)$  and  $(p_1, f^*)$  decide different information about  $\dot{d} \cap z$  for some  $z$ . In particular if we force with the  $f$ -parts under the term ordering, then we have that for every  $p$ , there are  $p_0, p_1 \leq p$  such that  $p_0$  and  $p_1$  force different values for  $\dot{d}$  at some  $z$ .

It is straightforward to see that in  $V[\mathbb{M} \restriction \alpha \times \mathbb{X} \times \mathbb{E}]$  extended by the poset of  $f$ -parts under the term ordering  $\sigma$  is preserved and  $(\mathbb{P}/\mathbb{P} \restriction \alpha)^2$  has the  $\sigma$ -cc. By induction, we construct  $\langle (p_i^0, p_i^1) \mid i < \sigma \rangle$  and  $\langle z_i \mid i < \sigma \rangle$  as follows. Suppose that for some  $j < \sigma$ , we have  $(p_i^0, p_i^1)$  and  $z_i$  for all  $i < j$ . Since the forcing of  $f$ -parts is  $< \sigma$ -distributive,  $\bigcup_{i < j} z_i$  is in  $\mathcal{P}_\sigma(\lambda)$  as computed in  $V[\mathbb{M} \restriction \alpha \times \mathbb{X} \times \mathbb{E}]$ . Hence we can find some  $p \in \mathbb{P}/\mathbb{P} \restriction \alpha$  which decides the value of  $\dot{d}$  up to it. Now apply the property from the previous paragraph to get  $p_j^0$  and  $p_j^1$  below  $p$ . This completes the construction. It follows that  $\langle (p_i^0, p_i^1) \mid i < \sigma \rangle$  is an antichain in  $(\mathbb{P}/\mathbb{P} \restriction \alpha)^2$ , a contradiction.  $\square$

From now on we assume for a contradiction that  $\dot{d}$  is a forced to be a new subset of  $\lambda$ . We use Claim 5.6 to finish the proof of the lemma. Let  $(p^*, f^*)$  witness the claim. Working in  $V$  we can assume that the trivial condition forces that the conclusion of the claim holds for  $(p^*, f^*)$ .

**Claim 5.7.** *Working in  $V$ , for any condition  $(e, x, \bar{p}, \bar{f}) \in \mathbb{E} \times \mathbb{X} \times \mathbb{M} \restriction \alpha$ , and any  $f \leq f^*$ , there are  $x' \leq x$ ,  $\bar{f}' \leq \bar{f}$ ,  $f_0, f_1 \leq f$  and an  $\mathbb{E} \times \mathbb{X} \times \mathbb{M} \restriction \alpha$ -name  $\dot{z}$  for an element of  $\mathcal{P}_\sigma(\mu)$  such that  $(e, x', \bar{p}, \bar{f})$  forces that  $(p^*, f_0)$  and  $(p_1, f_1)$  are in  $\mathbb{M}/\mathbb{M} \restriction \alpha$  and decide different information about  $\dot{d} \cap \dot{z}$ .*

*Proof.* We construct a predense set  $D$  in  $\mathbb{E} \times \mathbb{X} \times \mathbb{P} \restriction \alpha$  and a decreasing sequence of conditions  $(f_i^0, f_i^1)$  for  $i < \nu$  such that each condition in  $D$  forces that for some  $i < \nu$   $(p^*, f_i^0)$  and  $(p^*, f_i^1)$  are in  $\mathbb{M}/\mathbb{M} \restriction \alpha$  and decide different information about  $\dot{d} \cap z$  for some  $z$ . The witnessing name  $\dot{z}$  is obtained by collecting the information about

different elements  $z$  appearing in the construction. In advance of the construction we fix a wellordering  $\langle e_\beta \mid \beta < \nu \rangle$  of  $\mathbb{E}$ .

By induction on  $\beta < \nu$ , we construct  $(f_\beta^0, f_\beta^1)$  decreasing sequence in the term ordering and  $x_\beta$  decreasing in  $\mathbb{X}$ . We let  $x_0 = x$  and  $f_0^0 = f_0^1 = f$ . Now suppose that for some  $\beta < \nu$ , we have completed the construction for all  $\beta' \leq \beta$ . Let  $k$  be greatest such that  $\nu_k \leq \beta$  and write  $x_\beta$  as  $(\bar{x}_\beta, \hat{x}_\beta)$  in  $\mathbb{X}_{k+2} \times \mathbb{X}^{k+2}$ . For ease of notation we set  $e = e_\beta$ . To finish the successor step, we complete another induction. This time on ordinals  $i < \nu_{k+2}$ . We construct

- (1)  $(f_\beta^0(i), f_\beta^1(i))$
- (2)  $\bar{x}_\beta(i)$
- (3)  $\hat{x}_\beta(i)$
- (4)  $p_\beta(i)$
- (5)  $e_\beta(i)$  and
- (6)  $\dot{z}_\beta(i)$

where the construction will terminate at some successor step below  $\nu_{k+2}$ . The collection of  $(\bar{x}_\beta(i), p_\beta(i))$  for relevant  $i$  will form a maximal antichain in  $\mathbb{X}_{k+2} \times \mathbb{P} \restriction \alpha$ . The sequences  $\hat{x}_\beta(i)$  and  $(f_\beta^0(i), f_\beta^1(i))$  are decreasing in  $\mathbb{X}^{k+2}$  and  $\mathbb{C}^2$  respectively. Recall that  $\mathbb{C}$  is the poset of Mitchell like collapses for which  $\mathbb{M} = \mathbb{P} * \mathbb{C}^+$ . We set  $(\bar{x}_\beta(0), \hat{x}_\beta(0)) = (\bar{x}_\beta, \hat{x}_\beta)$  and  $(f_\beta^0(0), f_\beta^1(0)) = (f_\beta^0, f_\beta^1)$ . At some successor step  $i < \nu_{k+2}$ , we let  $(p, \bar{x})$  be incompatible with all  $(p_\beta(j), \bar{x}_\beta(j))$  for  $j \leq i$  if possible. Using Claim 5.6 and our assumption for a contradiction, we can find  $(f_\beta^0(i+1), f_\beta^1(i+1))$  below  $(f_\beta^0(i), f_\beta^1(i))$ ,  $e_\beta(i+1) \leq e$ , a condition  $(p_\beta(i+1), \bar{x}_\beta(i+1), \hat{x}_\beta(i+1))$  below  $(p, \bar{x}, \hat{x}_\beta(i))$  and a name  $\dot{z}_\beta(i+1)$  such that  $f_\beta^0(i+1) \restriction \alpha = f_\beta^1(i+1) \restriction \alpha$  and letting  $\bar{f}$  be this common value  $(p_\beta(i+1), \bar{f}, \bar{x}_\beta(i+1), \hat{x}_\beta(i+1), e_\beta(i+1))$  viewed as condition in  $\mathbb{M} \restriction \alpha \times \mathbb{X} \times \mathbb{E}$  forces that  $(p^*, f_\beta^0(i+1))$  and  $(p^*, f_\beta^1(i+1))$  force different values for  $\dot{d} \cap \dot{z}_\beta(i+1)$ . This finishes the successor step. At limits we can take lowerbounds for the decreasing sequences using the closure of the relevant forcing and leave the rest of the sequences undefined. The construction must halt at some successor step below  $\nu_{k+2}$ . If it halts at stage  $i$ , then we set  $(f_{\beta+1}^0, f_{\beta+1}^1) = (f_\beta^0(i), f_\beta^1(i))$  and  $x_{\beta+1}$  to be the condition formed by joining  $\bar{x}_\beta$  with  $\hat{x}_\beta(i)$ .

At limit stages of the induction on  $\beta$  take lowerbounds for each of the sequences. Note that for  $i \in [\nu_k, \nu_{k+1})$  we only decreased the part of the condition in  $\mathbb{X}$  which is in  $\mathbb{X}^{k+2}$ . This completes the induction on  $\beta$ . We let  $(f_0, f_1)$  and  $x'$  be lowerbounds for  $\langle (f_\beta^0, f_\beta^1) \mid \beta < \nu \rangle$  and  $\langle x_\beta \mid \beta < \nu \rangle$  respectively. Note that by the construction  $f_0 \restriction \alpha = f_1 \restriction \alpha$  and we call this common value  $\bar{f}'$ .

The following claim is straightforward.

**Claim 5.8.** *The set  $\{(p_\beta(i+1), e_\beta(i+1), \bar{x}_\beta(i+1) \smallfrown \hat{x}_\beta(i+1)) \mid i < \mu_{k+2}, \beta < \nu\}$  is predense in  $\mathbb{P} \restriction \alpha \times \mathbb{X} \times \mathbb{E}$  below  $(1_{\mathbb{P} \restriction \alpha}, x', 1_{\mathbb{E}})$ .*

Using the claim and choosing a some maximal antichain we can amalgamate the names  $\dot{z}_\beta(i+1)$  for  $\beta < \nu$  and relevant  $i$  into a single name  $\dot{z}$  for an element of  $\mathcal{P}_\sigma(\lambda)$ . Now it is clear from the construction that  $(f_0, f_1)$ ,  $x'$ ,  $\bar{f}'$  and  $\dot{z}$  satisfy the conclusion of the lemma.  $\square$

Using Claim 5.7, we build a binary tree of conditions  $\langle f_s \mid s \in 2^{<\rho} \rangle$  using an enumeration  $s_k$  of  $2^{<\rho}$  of ordertype less than  $\nu_0$  where if  $s_k$  is an initial segment of  $s_j$ , then  $k < j$ . We can ensure that there is a condition  $(\emptyset, x, \emptyset, \bar{f})$  which forces

that for all  $s \in 2^{<\rho}$ ,  $(p^*, f_{s \smallfrown 0})$  and  $(p^*, f_{s \smallfrown 1})$  decide different information about  $\dot{d} \cap \dot{z}_s$  for some name  $\dot{z}_s$ . We take a name  $\dot{z}^*$  which is forced to contain all elements of  $\mathcal{P}_\sigma(\mu)$  appearing in the construction.

Let  $\dot{b}$  be the canonical name for the first subset of  $\rho$  added by  $\mathbb{M}/\mathbb{M} \restriction \alpha$ . Using a standard construction on names, there is a lowerbound  $f^{**}$  for the sequence  $\langle f_{\dot{b} \restriction \alpha} \mid \alpha < \rho \rangle$ . Moreover we can assume that  $(p^*, f^{**})$  is forced to be in  $\mathbb{M}/\mathbb{M} \restriction \alpha$ . We pass to the extension by  $\mathbb{E} \times \mathbb{X} \times \mathbb{M} \restriction \alpha$  below the condition  $(\emptyset, x, \emptyset, \bar{f})$ .

We find an extension of  $(p^*, f^{**})$  which decides the value of  $\dot{d} \cap \dot{z}^*$  to be  $Y$ . This allows us to completely define  $\dot{b}$ , which is a contradiction.  $\dot{b} \restriction \alpha + 1$  is the unique  $s \in 2^{<\rho}$  such that  $(p^*, f^{**})$  forces that  $\dot{d} \cap \dot{z}_{s \restriction \alpha} = Y \cap \dot{z}_{s \restriction \alpha}$ .  $\square$

We remark that it is possible to weaken the assumptions of the lemma in a slight, but important way.

**Remark 5.9.** *We can weaken the condition  $\rho < \nu_0$  in Lemma 5.5 to the assertion that there is some  $k < \omega$  such that  $\nu_k > \rho$ . The only change required is in Claim 5.7. In particular, if we fix such a  $k$ , then we can assume that for  $\beta < \nu_k$ , all extensions in  $\mathbb{X}$  have the same part in  $\bar{\mathbb{X}}_{k+2}$ . At the end of the construction, we will have that  $x' \leq x$  and the piece of each condition in  $\bar{\mathbb{X}}_{k+2}$  is the same. This stronger condition allows us to complete the final argument, since the required extensions in  $\mathbb{X}$  are  $\nu_{k+2}$ -closed.*

## 6. THE FAILURE OF APPROACHABILITY IN THE FINAL MODEL

Let  $R$  be  $\mathbb{R}$ -generic. We give notation for the generic objects added by  $R$ . Let  $\langle x_n \mid n < \omega \rangle$  be the Prikry generic sequence. For all  $n < \omega$  we let  $\lambda_n = x_n \cap \kappa$  and for all  $i \leq \omega + 3$ ,  $\lambda_{n,i} = (\alpha \mapsto \alpha_i)(\lambda_n)$  where  $\alpha \mapsto \alpha_i$  is the function defined at the beginning of Section 2.

For  $n \geq 1$  we have the following generic objects induced by  $R$ :

- (1)  $Q_n = Q_n^0 \times Q_n^1$  which is generic for  $\mathbb{Q}^0(x_{n-1}, x_n) \times \mathbb{Q}^1(x_{n-1}, x_n)$ .
- (2)  $Q_n^0$  can be written as  $P_n^0 * S_n^0$  where  $P_n^0$  is generic for  $\text{Add}^V(\lambda_{n-1, \omega+2}, \lambda_n)$  and  $S_n^0$  is generic for  $\mathbb{C}^+(\text{Add}^V(\lambda_{n-1, \omega+2}, \lambda_n), \lambda_{n-1, \omega+3}, \lambda_n)$  over the extension by  $P_n^0$ .
- (3)  $Q_n^1$  can be written as  $P_n^1 * S_n^1$  where  $P_n^1$  is generic for  $\text{Add}^V(\lambda_{n-1, \omega+3}, \lambda_{n,1})$  and  $S_n^1$  is generic for  $\mathbb{C}^+(\text{Add}^V(\lambda_{n-1, \omega+3}, \lambda_{n,1}), \lambda_n, \lambda_{n,1})$ .
- (4) In a cardinal preserving extension, there are generics  $C_n^0, C_n^1$  which are generic for  $\mathbb{C}(\text{Add}^V(\lambda_{n-1, \omega+2}, \lambda_n), \lambda_{n-1, \omega+3}, \lambda_n)$  and  $\mathbb{C}(\text{Add}^V(\lambda_{n-1, \omega+3}, \lambda_{n,1}), \lambda_n, \lambda_{n,1})$  respectively.

Finally, we let  $Q_0$  be the induced generic for  $\text{Coll}(\omega, \lambda_{0, \omega})$ .

**Lemma 6.1.** *In  $V[G * H][R]$ ,  $\aleph_{\omega^2+1} \notin I[\aleph_{\omega^2+1}]$ .*

In joint work with Sinapova [24], we provided a sufficient condition for the failure of weak square in diagonal Prikry extensions. It is straightforward to check that  $\mathbb{R}$  is a *diagonal Prikry forcing* as in Definition 19 and also satisfies the hypotheses of Theorem 26 from that paper. Hence we have the failure of weak square in the extension. To prove the lemma above, we give a direct argument for the failure of approachability and note that the technique generalizes to give a metatheorem for the failure approachability in extensions by diagonal Prikry type forcing.



*Proof.* Recall that  $\vec{K}$  is  $\mathbb{Q}^\omega$ -generic over  $V[G * H]$ . Note that in  $V[G * H][\vec{K}]$ ,  $\kappa$  is  $\kappa_{\omega+1}$ -supercompact as witnessed by  $U^*$  the measure on  $\mathcal{P}_\kappa(\kappa_{\omega+1})$  derived from  $j$  and  $\mathbb{R}/I$  is  $\kappa_{\omega+1}$ -cc where  $I$  is the induced  $\mathbb{Q}^\omega/\text{fin}$ -generic object. By Remark 5.1, it is enough to show that approachability fails when we force with  $\mathbb{R}/I$  over  $V[G * H]^{\mathbb{Q}^\omega}$ . Assume for a contradiction that  $\langle \dot{a}_\alpha \mid \alpha < \kappa_{\omega+1} \rangle$  is a name for a sequence witnessing approachability. Let  $k : V[G * H][\vec{K}] \rightarrow M$  be the ultrapower by  $U^*$ . By the construction of  $\mathbb{R}$ , we can choose a condition  $r \in k(\mathbb{R})$  which forces that  $\kappa_{\omega+1} = \omega_1$ . We let  $\gamma = \sup k''\kappa_{\omega+1}$ .

It follows that  $r$  forces that  $\gamma$  is approachable with respect to  $k(\langle \dot{a}_\alpha \mid \alpha < \kappa_{\omega+1} \rangle)$ . So there is a  $k(\mathbb{R}/I)$ -name  $\dot{A}$  for a subset of  $\gamma$  all of whose initial segments are enumerated on the sequence  $k(\langle \dot{a}_\alpha \mid \alpha < \kappa_{\omega+1} \rangle)$  before stage  $\gamma$ . By standard arguments we can assume that  $\dot{A}$  is forced to be closed. Since  $\text{cf}(\gamma) = \kappa_{\omega+1}$  and it is forced by  $r$  that every club subset of  $\kappa_{\omega+1}$  contains a club from the ground model, there is a club subset  $B$  of  $\gamma$  which is forced to be a subset of  $\dot{A}$ . We let  $C = \{\alpha \mid k(\alpha) \in B\}$ . It is straightforward to see that  $C$  is  $<\kappa$ -club in  $\kappa_{\omega+1}$ . Let  $\eta$  be the  $\kappa_\omega$ -th element in an increasing enumeration of  $C$ .

We can assume that there is an index  $\bar{\gamma} < \kappa_{\omega+1}$  such that  $r$  forces  $\dot{A} \cap k(\eta)$  is enumerated before stage  $k(\bar{\gamma})$  in  $k(\langle \dot{a}_\alpha \mid \alpha < \kappa_{\omega+1} \rangle)$ . Now for every  $x \subseteq C \cap \eta$  of ordertype  $\omega$ , there is a condition  $r_x \in \mathbb{R}/I$  which forces that  $x \subseteq \dot{a}_\alpha$  for some  $\alpha < \bar{\gamma}$ . Note that for a given  $x$ ,  $r$  witnesses  $k$  applied to this statement.

By the chain condition of  $\mathbb{R}/I$ , we can find a condition which forces that for  $\kappa_{\omega+1}$  many  $x$ ,  $r_x$  is in the generic. This is impossible, since we can assume that each  $\dot{a}_\alpha$  is forced to have ordertype less than  $\kappa$  and hence  $|\bigcup_{\alpha < \bar{\gamma}} \mathcal{P}(\dot{a}_\alpha)| \leq \kappa$ .  $\square$

**Remark 6.2.** *Note that none of the specific properties of  $\mathbb{R}$  are used in the proof above and hence the assumptions of Theorem 26 of [24] are enough to show the failure of approachability.*

Next we take care of the successors of singulars below  $\aleph_{\omega^2}$ .

**Lemma 6.3.** *There is a condition of length 0 in  $\mathbb{R}$  which forces that for all  $n \geq 1$ , there is a bad scale of length  $\aleph_{\omega \cdot n + 1}$  in some product of regular cardinals.*

It follows that for all  $n < \omega$ ,  $\aleph_{\omega \cdot n + 1} \notin I[\aleph_{\omega \cdot n + 1}]$ .

*Proof.* Working in  $V[G * H]$ , fix a scale  $\vec{f}$  of length  $\kappa^{+\omega+1}$  in some product of regular cardinals. By standard arguments there is a  $U_0$ -measure one set  $A$  such that for all  $\delta \in A$ , there are stationarily many bad points for  $\vec{f}$  of cofinality  $\delta_{\omega+1}$ . This is absolute to  $M[G * H]$ . It follows that  $\kappa$  is in the set given by  $j$  applied to  $B = \{\gamma \mid \text{there is a scale of length } \gamma_{\omega+1} \text{ such that for all } \delta \in A \cap \gamma \text{ there are stationarily many bad points of cofinality } \delta_{\omega+1}\}$ . It follows that  $B \in U_0$ .

Now for each  $i \geq 1$ , there is a  $U_i$ -measure one set  $A_i$  of  $x$  such that  $\kappa_x \in A \cap B$ . For any  $x \prec y$  such that for some  $i < i'$   $x \in A_i$  and  $y \in A_{i'}$ , we have arranged the following property. Since  $\kappa_x \in A \cap B \cap \kappa_y$  and by the choice of  $A$  and  $B$ , there are stationarily many bad points of cofinality  $\kappa_{x, \omega+1}$  for some scale of length  $\kappa_{y, \omega+1}$ .

The condition required for the lemma is any condition length 0 whose measure one sets are contained in the  $A_i$ . Work below such a condition and fix  $n \geq 1$ . Let  $p$  be a condition of length  $n + 1$  and  $\vec{f}$  be a scale of length  $\kappa_{x_n^p, \omega+1}$  such that there is a stationary set  $S$  of bad points of cofinality  $\kappa_{x_0^p, \omega+1}$ . By Corollary 4.15, it is enough to show that  $\vec{f}$  remains a scale with stationary set of bad points  $S$  in the

model  $V[G * H][\prod_{i < n} Q_i]$ . The forcing to add  $\prod_{i < n} Q_i$  is small relative to  $\kappa_{x_n^p, \omega+1}$  and hence it is easy to see that  $\vec{f}$  remains a scale and  $S$  remains stationary. So it is enough to show that every point in  $S$  is still bad for  $\vec{f}$  in the extension.

In this extension  $\kappa_{x_0^p, \omega+1}$  becomes  $\aleph_1$  via  $\text{Coll}(\omega, \kappa_{x_0^p, \omega})$  and every  $\aleph_1$ -sequence of ordinals in  $V[G * H][\prod_{i < n} Q_i]$  is in the extension by this collapse. Now a standard argument shows that for every  $\delta \in S$  and every unbounded subset  $A$  of  $\delta$  in the extension there is an unbounded subset of  $A$  in  $V[G * H]$ . So if  $A$  witnesses that  $\delta \in S$  is good in the extension, then there is an unbounded subset of  $A$  witnessing that  $\delta$  is good in  $V[G * H]$ . This is impossible, so we must have the every point in  $S$  is bad in  $V[G * H][\prod_{i < n} Q_i]$ . This completes the proof.  $\square$

For the cardinals which are not successors of singulars, we apply the scheme from the previous section. Throughout the proofs of the following lemmas, we omit the straightforward but tedious verification that our scheme from Section 5 applies and that the hypotheses of our preservation lemmas hold.

**Lemma 6.4.** *In  $V[G * H][R]$ ,  $\aleph_{\omega^2+2} \notin I[\aleph_{\omega^2+2}]$*

*Proof.* Note that by Remark 5.1 it is enough to show the conclusion in an outer model with the same cardinals up to  $\aleph_{\omega^2+2}$ . So we show that it holds in  $V[G * H][\vec{K}][\vec{R}]$  where  $\vec{K}$  is generic for  $\mathbb{Q}^\omega$  and  $\vec{R}$  is generic for  $\bar{\mathbb{R}}$  as defined in the extension by  $\vec{K}$ . Since  $\bar{\mathbb{R}}$  is  $\kappa_\omega$ -centered in  $V[G * H][\vec{K}]$ , by Theorem 5.2 it is enough to show that  $\kappa_{\omega+2} \notin I[\kappa_{\omega+2}]$  in  $V[G * H][\vec{K}]$ .

We write  $H$  as  $P_0 \times P_1 * \prod_{i \leq \omega+1} C_i^+$  where  $P_1$  is generic for  $\mathbb{P}_1 \times \text{Add}(\kappa_1, \theta^+ \setminus \theta) \simeq \text{Add}(\kappa_1, \theta^+)$ . Again by Remark 5.1, it is enough to show  $\kappa_{\omega+2} \notin I[\kappa_{\omega+2}]$  in the extension

$$V[\vec{K}][G][C_{\omega+1}][\prod_{i < \omega} C_i][P_0][P_1 \restriction [\kappa_{\omega+2}, \theta^+]][P_1 \restriction \kappa_{\omega+2} * C_\omega^+].$$

Note that in  $V[\vec{K}][G][C_{\omega+1}]$  we have that  $\kappa_{\omega+2}$  is still Mahlo. So in this model we apply the scheme from Section 5 and Lemma 5.4 where

- (1)  $\lambda = \kappa_2$ ,
- (2)  $\mathbb{M}$  is the forcing to add  $P_1 \restriction \kappa_{\omega+2} * C_\omega^+$ ,
- (3)  $\bar{\mathbb{X}}$  is the forcing to add  $P_0 \times C_0 \times P_1 \restriction [\kappa_{\omega+2}, \theta^+]$ ,
- (4)  $\bar{\mathbb{X}}$  is the forcing to add  $\prod_{0 < i < \omega} C_i$

$\square$

**Lemma 6.5.** *In  $V[G * H][R]$ ,  $\aleph_{\omega^2+3} \notin I[\aleph_{\omega^2+3}]$ .*

The proof is similar to the proof of the previous lemma with some changes of the details.

*Proof.* Again by Theorem 5.2 and Claim 4.6 it is enough to show that  $\kappa_{\omega+3} \notin I[\kappa_{\omega+3}]$  in  $V[G * H][\vec{K}]$ . By the proof of Claim 4.6, there is a cardinal preserving outer model of  $V[G * H][\vec{K}]$  where we have decomposed  $\vec{K}$  as  $K^0 \times K^1$  which is generic for the product of  $\kappa_{\omega+3}$ -closed forcing and  $\kappa_{\omega+3}$ -cc forcing both taken from  $V$ . The  $\kappa_{\omega+3}$ -cc forcing is just  $\text{Add}(\kappa_{\omega+2}, \eta)$  for some  $\eta$ .

As in the previous lemma we pass to an outer model where we have decomposed  $H$ . In particular it is enough to prove that  $\kappa_{\omega+3} \notin I[\kappa_{\omega+3}]$  in the model

$$V[K^0][G][P_0 \times \prod_{i \leq \omega} C_i][K^1][P_1 \restriction [\kappa_{\omega+3}, \kappa_{\omega+3}^+]][P_1 \restriction \kappa_{\omega+3} * C_{\omega+1}^+].$$

Note that in  $V[K^0][G]$ ,  $\kappa_{\omega+3} = \theta$  is still Mahlo. So in this model we apply the scheme from Section 5 and Lemma 5.4 where

- (1)  $\lambda = \kappa_{\omega+2}$ ,
- (2)  $\mathbb{M}$  is the forcing to add  $P_1 \restriction \kappa_{\omega+3} * C_{\omega+1}^+$ ,
- (3)  $\bar{\mathbb{X}}$  as the forcing to add  $P_0 \times P_1 \restriction [\kappa_{\omega+3}, \kappa_{\omega+3}^+) \times \prod_{i \leq \omega} C_i$ , and
- (4)  $\hat{\mathbb{X}}$  as the forcing to add  $K^1$ .

□

**Lemma 6.6.** *In  $V[G * H][R]$  for each successor  $\tau$  of a regular cardinal with  $\tau \in [\aleph_2, \aleph_{\omega^2})$ ,  $\tau \notin I[\tau]$ .*

*Proof.* There are a few cases based on how close  $\tau$  is to the collapses between the Prikry points. Some cardinals we must treat individually and others we can treat uniformly.

First we assume that  $\tau = \aleph_2$ . Note that in  $V[G * H][R]$ ,  $\lambda_{0,\omega+1} = \aleph_1$  and  $\lambda_{0,\omega+2} = \aleph_2$ . Notice that any sequence witnessing that  $\aleph_2 \in I[\aleph_2]$  is in the extension  $V[G \restriction \lambda_0 + 1][Q_0 \times Q_1]$ . Recall that  $Q_0$  is generic for  $\text{Coll}(\omega, \lambda_{0,\omega})$  and  $Q_1$  is generic for  $\lambda_{0,\omega+2}$ -closed forcing from  $V$ .

We let  $P_0 \times P_1$  be generic for  $\mathbb{P}_0(\lambda_0) \times (\mathbb{P}_1(\lambda_0) \times \text{Add}(\lambda_{0,1}, \theta_{\lambda_0}^+ \setminus \theta_{\lambda_0}))$  and for all  $i \leq \omega + 1$ ,  $C_i^+$  be the generic for  $\mathbb{C}_i^+(\lambda_0)$ .

By Remark 5.1, it is enough to show that  $\aleph_2 \notin I[\aleph_2]$  in the extension

$$V[Q_1][G \restriction \lambda_0][C_{\omega+1}][\prod_{i < \omega} C_i][P_0][P_1 \restriction [\lambda_{0,\omega+2}, \theta_{\lambda_0}^+]][Q_0][P \restriction \lambda_{0,\omega+2} * C_{\omega}^+].$$

Note that in  $V[Q_1][G \restriction \lambda_0][C_{\omega+1}]$  we have that  $\lambda_{0,\omega+2}$  is Mahlo. So in this model we apply the scheme from Section 5 and Lemma 5.5 and Remark 5.9 where

- (1)  $\mathbb{M}$  is the forcing to add  $P \restriction \lambda_{0,\omega+2} * C_{\omega}^+$ ,
- (2)  $\mathbb{E}$  is  $\text{Coll}(\omega, \lambda_{0,\omega})$  which adds  $Q_0$ ,
- (3)  $\mathbb{X}$  is the forcing to add  $\prod_{i < \omega} C_i \times P_0 \times P_1 \restriction [\lambda_{0,\omega+2}, \theta_{\lambda_0}^+)$  and
- (4) the sequence  $\nu_k$  for  $k < \omega$  is the sequence  $\lambda_{0,k}$  for  $k < \omega$ .

It follows that  $\aleph_2 \notin I[\aleph_2]$  in the extension.

Next we assume that  $\tau = \lambda_{n,\omega+2}$  for some  $n < \omega$ . This is similar to previous case, but easier. Any sequence witnessing that  $\tau \in I[\tau]$  in  $V[G * H][R]$  is in the extension  $V[G \restriction \lambda_n + 1][\prod_{k \leq n} Q_k][Q_{n+1}]$ . As before we let  $P_0 \times P_1$  be generic for  $\mathbb{P}_0(\lambda_n) \times (\mathbb{P}_1(\lambda_n) \times \text{Add}(\lambda_{n,1}, \theta_{\lambda_n}^+ \setminus \theta_{\lambda_n}))$  and  $\prod_{k \leq \omega+1} C_k^+$  be generic for  $\mathbb{C}^+(\lambda_n)$ .

By Remark 5.1, it is enough to show that  $\tau \notin I[\tau]$  in the model

$$V[G \restriction \lambda_n][C_{\omega+1}][Q_{n+1}][\prod_{k \leq n} Q_k][C_0][P_0][P_1 \restriction [\lambda_{n,\omega+2}, \theta_{\lambda_n}^+]][\prod_{0 < i < \omega} C_i][P_1 \restriction \lambda_{n,\omega+2} * C_{\omega}^+]$$

Note that in  $V[G \restriction \lambda_n][C_{\omega+1}][Q_{n+1}]$ ,  $\lambda_{n,\omega+2}$  is still Mahlo. So in this model we apply the scheme from Section 5 and Lemma 5.4 where

- (1)  $\lambda$  is  $\lambda_{n,2}$ ,
- (2)  $\mathbb{M}$  is the forcing to add  $P \restriction \lambda_{n,\omega+2} * C_{\omega}^+$ ,
- (3)  $\bar{\mathbb{X}}$  is the forcing to add  $P_0 \times P_1 \restriction [\lambda_{n,\omega+2}, \theta_{\lambda_n}^+) \times C_0$  and
- (4)  $\hat{\mathbb{X}}$  is the forcing to add  $\prod_{0 < i < \omega} C_i$ .

This completes the argument that  $\lambda_{n,\omega+2} \notin I[\lambda_{n,\omega+2}]$  for  $0 < n < \omega$ .

Suppose that  $\tau = \lambda_{n,\omega+3}$  for  $n < \omega$ . Any sequence witnessing  $\tau \in I[\tau]$  in  $V[G * H][R]$  is in the extension by  $V[G \restriction \lambda_{n+1}][\prod_{k \leq n+1} Q_k]$ . Recall that by passing

to an outer model, we can decompose  $Q_{n+1}$  as  $P_{n+1}^0 \times C_{n+1}^0 \times P_{n+1}^1 \times C_{n+1}^1$ . As before we let  $P_0 \times P_1$  be generic for  $\mathbb{P}_0(\lambda_n) \times (\mathbb{P}_1(\lambda_n) \times \text{Add}(\lambda_{n,1}, \theta_{\lambda_n}^+ \setminus \theta_{\lambda_n}))$  and  $\prod_{k \leq \omega+1} C_k^+$  be generic for  $\mathbb{C}^+(\lambda_n)$ .

By Remark 5.1 it is enough to show that  $\tau \notin I[\tau]$  in the model

$$V[G \restriction \lambda_n][\prod_{k \leq n} Q_k][P_{n+1}^1 \times C_{n+1}^1 \times C_{n+1}^0][P_{n+1}^0][P_0][P_1 \restriction [\theta_{\lambda_n}, \theta_{\lambda_n}^+]][\prod_{k \leq \omega} C_k][P_1 \restriction \lambda_{n,\omega+3} * C_{\omega+1}^+]$$

We have that  $\lambda_{n,\omega+3}$  is Mahlo in the model  $V[G \restriction \lambda_n][\prod_{k \leq n} Q_k][P_{n+1}^1 \times C_{n+1}^1 \times C_{n+1}^0]$ . So we apply the scheme from Section 5 and Lemma 5.4 in this model where

- (1)  $\lambda = \lambda_{n,\omega+2}$ ,
- (2)  $\mathbb{M}$  is the forcing to add  $P \restriction \lambda_{n,\omega+3} * C_{\omega+1}^+$ ,
- (3)  $\bar{\mathbb{X}}$  is the forcing to add  $P_0 \times P_1 \restriction [\theta_{\lambda_n}, \theta_{\lambda_n}^+] \times \prod_{k \leq \omega} C_k$  and
- (4)  $\hat{\mathbb{X}}$  is the forcing to add  $P_{n+1}^0$ .

This completes the argument that  $\lambda_{n,\omega+3} \notin I[\lambda_{n,\omega+3}]$  for all  $n < \omega$ .

Next we assume that  $\tau = \lambda_n$  for  $n \geq 1$ . Any sequence witnessing that  $\tau \in I[\tau]$  in  $V[G * H][R]$  is in  $V[G \restriction \lambda_n + 1][\prod_{i \leq n} Q_i]$ . As before by passing to an outermodel, we can decompose  $Q_n$  as  $P_n^0 * S_n^0 \times P_n^1 \times C_n^1$ . By Remark 5.1, it is enough to show that there are no such sequences in the outer model

$$V[C_n^1][Y][Z][G \restriction \lambda_{n-1} + 1][\prod_{i \leq n-1} Q_i][P_n^1][P_n^0 * S_n^0]$$

where  $Y$  is generic for  $\mathcal{A}(\mathbb{A} \restriction \lambda_n, \mathbb{A}(\lambda_n))$  and  $Z$  is generic for  $\mathcal{A}(\mathbb{A}) \restriction [\lambda_{n-1} + 1, \lambda_n)$ .

By Fact 2.6,  $\lambda_n$  is still Mahlo in  $V[C_n^1][Y][Z]$ . Moreover, the computation of  $\mathbb{Q}_n^0 = \mathbb{M}(\lambda_{n-1,\omega+2}, \lambda_{n-1,\omega+3}, \lambda_n)$  is the same in  $V$  and  $V[C_n^1][Y][Z]$ , since the forcing to add  $Z$  is closed beyond the first inaccessible above  $\lambda_{n-1,\omega+3}$ .

So in this model we apply the scheme from Section 5 and Lemma 5.4 where

- (1)  $\lambda = \lambda_{n-1,\omega+3}$ .
- (2)  $\mathbb{M}$  is the forcing to add  $P_n^0 * S_n^0$ ,
- (3)  $\bar{\mathbb{X}}$  is the forcing to add  $G \restriction (\lambda_{n-1} + 1) \times \prod_{i \leq n-1} Q_i$
- (4)  $\hat{\mathbb{X}}$  is the forcing to add  $P_n^1$  and

This finishes the proof that for all  $n \geq 1$ ,  $\lambda_n \notin I[\lambda_n]$  in  $V[G * H][R]$ .

Next we assume that  $\tau = \lambda_{n,1}$  for some  $n \geq 1$ . Any sequence witnessing that  $\tau \in I[\tau]$  is in the model  $V[G \restriction \lambda_n + 1][\prod_{i \leq n} Q_i]$ . By Remark 5.1 it is enough to show that  $\tau \notin I[\tau]$  in the model

$$V[Y][G \restriction \lambda_n][P_n^0][C_n^0][Y_{P_0}][P_n^1 * S_n^1]$$

where  $Y$  is generic for the poset of  $\mathbb{A} \restriction \lambda_n$ -names for elements of  $\mathbb{P}_1(\lambda_n) \times \text{Add}(\lambda_{n,1}, \theta_{\lambda_n}^+ \setminus \theta_{\lambda_n}) \times \mathbb{C}(\lambda_n)$  and  $Y_{P_0}$  is generic for the poset of  $\mathbb{A} \restriction \lambda_n$ -names for elements of  $\mathbb{P}_0(\lambda_n)$ . By Fact 2.5, we can take the forcing to add  $Y_{P_0}$  to be  $\text{Add}(\lambda_{n,0}, \lambda_{n,2})$  as computed in  $V$ .

Note that  $\lambda_{n,1}$  is still Mahlo in  $V[Y]$ . So we apply the scheme from Section 5 and Lemma 5.4 in this model where

- (1)  $\lambda = \lambda_n$ ,
- (2)  $\mathbb{M}$  is the forcing to add  $P_n^1 * S_n^1$ ,
- (3)  $\bar{\mathbb{X}}$  is the forcing to add  $G \restriction \lambda_n \times P_n^0 \times C_n^0$  and
- (4)  $\hat{\mathbb{X}}$  is the forcing to add  $Y_{P_0}$ .

This finishes the proof that  $\lambda_{n,1} \notin I[\lambda_{n,1}]$  for  $n \geq 1$ .

Next we assume that  $\tau = \lambda_{n,2}$  for  $n \geq 1$ . Any sequence witnessing that  $\tau \in I[\tau]$  in  $V[G * H][R]$  is in  $V[G * H][\prod_{k \leq n} Q_k]$ . By Remark 5.1, it is enough to show that  $\tau \notin I[\tau]$  in the model

$$V[G \restriction \lambda_n][\prod_{1 \leq i \leq \omega+1} C_i][\prod_{k \leq n} Q_k][P_1][P_0 * C_0^+].$$

Note that  $\lambda_{n,2}$  is still Mahlo in  $V[G \restriction \lambda_n][\prod_{1 \leq i \leq \omega+1} C_i]$ . Hence we apply the scheme from Section 5 and Lemma 5.4 in this model where

- (1)  $\lambda = \lambda_{n,1}$ ,
- (2)  $\mathbb{M}$  is the forcing to add  $P_0 * C_0^+$ ,
- (3)  $\mathbb{X}$  is the forcing to add  $\prod_{k \leq n} Q_k$  and
- (4)  $\hat{\mathbb{X}}$  is the forcing to add  $P_1$ .

Next we assume that  $\tau = \lambda_{n,i+1}$  for  $0 < n < \omega$  and  $2 \leq i < \omega$ . Any sequence witnessing that  $\tau \in I[\tau]$  in  $V[G * H][R]$  is in  $V[G \restriction \lambda_n + 1][\prod_{k \leq n} Q_k]$ . By Remark 5.1, it is enough to show that  $\tau \notin I[\tau]$  in the model

$$V[G \restriction \lambda_n][\prod_{k \geq i} C_k][\prod_{k \leq n} Q_k][\prod_{k < i-1} C_k][P_0][P_1 \restriction [\lambda_{n,i+1}, \theta_{\lambda_n}^+]][P_1 \restriction \lambda_{n,i+1} * C_{i-1}^+].$$

Note that in  $V[G \restriction \lambda_n][\prod_{k \geq i} C_k]$ ,  $\lambda_{n,i+1}$  is still Mahlo. Hence we apply the scheme from Section 5 and Lemma 5.4 in this model where

- (1)  $\lambda = \lambda_{n,i}$ ,
- (2)  $\mathbb{M}$  is the forcing to add  $P \restriction \lambda_{n,i+1} * C_{i-1}^+$ ,
- (3)  $\mathbb{X}$  is the forcing to add  $\prod_{k \leq n} Q_k \times \prod_{k < i-1} C_k \times P_0 \times P_1 \restriction [\lambda_{n,i+1}, \theta_{\lambda_n}^+]$  and
- (4)  $\hat{\mathbb{X}}$  is the trivial forcing.

This finishes the proof that for  $0 < n < \omega$  and  $2 \leq i < \omega$ ,  $\lambda_{n,i+1} \notin I[\lambda_{n,i+1}]$  and with it the proof of Lemma 6.6  $\square$

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